The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 2009 Problems

1. Let ABC be an acute-angled triangle, and let P and Q be points on sides AC and BC, respectively, such that $APQB$ is a cyclic quadrilateral. Let R be the point such that $PR \perp AC$ and $QR \perp BC$.

Prove that $CR \perp AB$.

2. Consider rectangular rooms whose side lengths are integers > 1. Tile the floors of such rooms with unit square tiles. *Edge tiles* are those next to the walls of the room (including corner tiles). The remaining tiles are *interior tiles*.

A pair (R, S) of rooms (R may have the same dimensions as S is *compatible* if

(i) the number of edge tiles in $R =$ the number of interior tiles in S, and
(ii) the number of edge tiles in $S =$ the number of interior tiles in R.

For a compatible pair (R, S), let $m(R, S)$ be the minimal length of an edge occurring in R or S.

Determine the largest value $m(R, S)$ can take.

3. The polynomials $x^2 + x$ and $x^2 + 2$ are written on a white board. Sue is allowed to write on this board the sum, the difference or the product of any two polynomials already on the board. She repeats this process as many times as she likes.

Can Sue ever write the polynomial x on the board?

4. Let $a_a, a_2, \ldots, a_m \in \mathbb{Z}$, where $m \geq 3$, and let $N = \dfrac{m(m+1)}{2}$.

Prove that there is an integer k such that none of the integers $a_i + a_j - k$ is divisible by N, for all pairs of integers (i, j) such that $1 \leq i, j \leq m$.

5. A certain country has a finite number of towns, and all the distances between the towns are different. Each town is connected to its nearest neighbour by a straight road, and there are no other roads in the country.

(a) Prove that no two roads cross each other.
(b) Prove that there is no circuit within this road network.

6. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(n) = \begin{cases} n - 9, & \text{if } n > 2009 \\ f\big(f(n + 1009)\big), & \text{if } n \leq 2009. \end{cases}$$

7. Let I be the incentre of $\triangle ABC$ in which $AC \neq BC$. Let Γ be the circle passing through A, I and B. Suppose Γ intersects the line AC at A and X and intersects the line BC at B and Y.

Show that $AX = BY$.

(The *incentre* of a triangle is the intersection of its angle bisectors.)

8. Let ABC be a triangle, and let X, Y, Z be points on the sides BC, CA, AB, respectively. Let T be the area of $\triangle XYZ$, and T_1, T_2, T_3 be the areas of $\triangle AYZ$, $\triangle BZX$, $\triangle CXY$, respectively.

Prove that $\quad \dfrac{3}{T} \leq \dfrac{1}{T_1} + \dfrac{1}{T_2} + \dfrac{1}{T_3}$.