

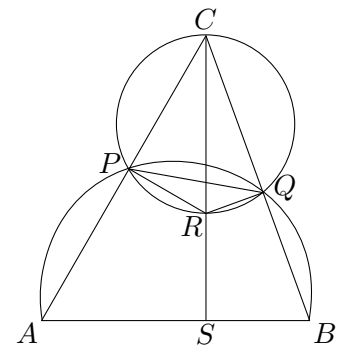
AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 2009 Problems
with Some Solutions

1. Let ABC be an acute-angled triangle, and let P and Q be points on sides AC and BC , respectively, such that $APQB$ is a cyclic quadrilateral. Let R be the point such that $PR \perp AC$ and $QR \perp BC$. Prove that $CR \perp AB$.

Solution. Let S be the intersection of CR and AB .

$$\begin{aligned} \angle CPR &= \angle CQR = 90^\circ, && \text{(given)} \\ \therefore \angle CPR \text{ and } \angle CQR &\text{ are supplementary} \\ \therefore QCPR &\text{ is a cyclic quadrilateral} \\ \angle CAS = \angle CAB &= 180^\circ - \angle PQB, && \text{since } APQB \text{ is a cyclic} \\ &&& \text{quadrilateral} \\ &= \angle CQP, && \text{since } \angle CQB \text{ is a straight} \\ &&& \text{angle} \\ &= \angle CRP, && \text{(angles in circumcircle of} \\ &&& \text{ } QCPR \text{ standing on } CP) \\ \angle ACS &= \angle RCP, && \text{(same angle)} \\ \therefore \triangle CAS &\sim \triangle CRP, && \text{by the AA Rule} \\ \therefore \angle CSA &= \angle CPR = 90^\circ \\ \therefore CS &= CR \perp AB \end{aligned}$$



2. Consider rectangular rooms whose side lengths are integers > 1 . Tile the floors of such rooms with unit square tiles. *Edge tiles* are those next to the walls of the room (including corner tiles). The remaining tiles are *interior tiles*.

A pair (R, S) of rooms (R may have the same dimensions as S) is *compatible* if

- (i) the number of edge tiles in R = the number of interior tiles in S , and
- (ii) the number of edge tiles in S = the number of interior tiles in R .

For a compatible pair (R, S) , let $m(R, S)$ be the minimal length of an edge occurring in R or S . Determine the largest value $m(R, S)$ can take.

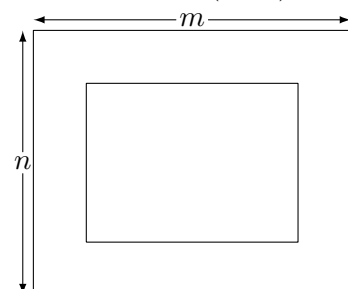
Solution. Suppose that the largest value of $m(R, S)$ is attained for pair (A, B) and suppose

$$\begin{aligned} x, y &= \text{dimensions of } A, \text{ and} \\ u, v &= \text{dimensions of } B. \end{aligned}$$

Also, without loss of generality, assume $x \leq y$, $u \leq v$ and $x \leq u$, so that $x = m(A, B)$.

In general, for a room with dimensions m, n ,

$$\begin{aligned} \text{number of edge tiles} &= 2m + 2n - 4, \text{ and} \\ \text{number of interior tiles} &= (m - 2)(n - 2) \\ &= mn - (2m + 2n - 4). \end{aligned}$$



Now A, B being compatible implies that

$$2x + 2y - 4 = (u - 2)(v - 2), \quad \text{by condition (i)} \quad (1)$$

$$(x - 2)(y - 2) = 2u + 2v - 4, \quad \text{by condition (ii)} \quad (2)$$

$$\therefore xy = uv, \quad (1) + (2)$$

$$\therefore v = \frac{x}{u} \cdot y$$

$$\leq y, \quad \text{since } x \leq u \implies \frac{x}{u} \leq 1$$

$$\therefore x \leq u \leq v \leq y. \quad (3)$$

Rearranging (2), we have:

$$xy = 2x + 2y + 2u + 2v - 8 \quad (4)$$

$$\leq 8y - 8, \quad \text{using (3)}$$

$$\therefore (8 - x)y \geq 8 \quad (5)$$

$$\therefore x \leq 7.$$

Suppose that $x = 7$. Then by (4),

$$7y = 14 + 2y + 2u + 2v - 8$$

$$5y = 6 + 2u + 2v$$

$$\leq 6 + 4y, \quad \text{using (3)}$$

$$\therefore y \leq 6.$$

But (5) (with $x = 7$) implies $y \geq 8$. So, we have a contradiction. Hence $x < 7$.

For $x = 6$, and with $u = 6 = x$ and $y = 8 = v$, so that A, B have the same dimensions, and consequently must also have equal numbers of edge and exterior tiles, namely 24. Thus $m(R, S) = 6$ is attainable.

Therefore the maximum value of $m(R, S)$ is 6.

3. The polynomials $x^2 + x$ and $x^2 + 2$ are written on a white board. Sue is allowed to write on this board the sum, the difference or the product of any two polynomials already on the board. She repeats this process as many times as she likes.

Can Sue ever write the polynomial x on the board?

Solution. Define a polynomial $p(x)$ to have Property P if

$$6 \mid p(2).$$

Observe that for

$$f(x) = x^2 + x \text{ and } g(x) = x^2 + 2,$$

we have

$$6 \mid f(2) \text{ and } 6 \mid g(2),$$

since $f(2) = 6 = g(2)$. So each of the initial polynomials has Property P .

Lemma. If f, g on board and $6 \mid f(2)$ and $6 \mid g(2)$ then $6 \mid (f + g)(2)$, $6 \mid (f - g)(2)$, and $6 \mid (f \cdot g)(2)$.

Proof. This follows immediately from the definitions of $f + g$, $f - g$ and $f \cdot g$:

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

So, for example, since $6 \mid f(2)$ and $6 \mid g(2)$, we have $6 \mid (f(2) + g(2)) = (f + g)(2)$. \square

Put another way, the Lemma says:

If two chosen polynomials on the board have Property P then the resultant polynomial has Property P .

Thus, it follows, by induction, that at every stage, every polynomial on the board has Property P . Suppose $h(x) = x$ is written on the board at some stage. Then it must have Property P . But $h(2) = 2$ and $6 \nmid 2$. So $h(x)$ does not have Property P (a contradiction).

Thus Sue can never write x on the board.

4. Let $a_1, a_2, \dots, a_m \in \mathbb{Z}$, where $m \geq 3$, and let $N = \frac{m(m+1)}{2}$.

Prove that there is an integer k such that none of the integers $a_i + a_j - k$ is divisible by N , for all pairs of integers (i, j) such that $1 \leq i, j \leq m$.

5. A certain country has a finite number of towns, and all the distances between the towns are different. Each town is connected to its nearest neighbour by a straight road, and there are no other roads in the country.

- (a) Prove that no two roads cross each other.
 (b) Prove that there is no circuit within this road network.

Solution.

- (a) Suppose for a contradiction there are towns A, B, C, D such that

C is the nearest neighbour of A and
 D is the nearest neighbour of B

so that there are roads AC and BD , and that these roads intersect at I .

Then, since all the distances between towns are different,

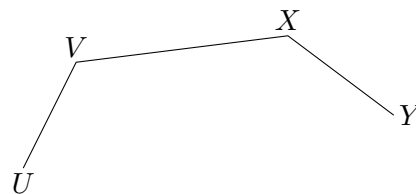
$$\begin{aligned} AC &< AD \text{ and} \\ BD &< BC \\ \therefore AC + BD &< AD + BC \end{aligned}$$

However, by the Triangle Inequality applied to $\triangle AID$ and $\triangle BIC$,

$$\begin{aligned} AI + ID &\geq AD \text{ and} \\ BI + IC &\geq BC \\ \therefore AC + BD &= (AI + IC) + (BI + ID) \geq AD + BC, \end{aligned}$$

and so we have a contradiction. Therefore no roads intersect.

- (b) Suppose for a contradiction there is a circuit in the network and the longest road in the circuit connects towns V and X , and towns U and Y (possibly the same town, in the case the circuit is a triangle), with U connected to V and X connected to Y . There is a unique longest road since all the distances between towns are different.



Now either V is connected to X because X is V 's nearest neighbour (which implies $VX < VU$) or because V is X 's nearest neighbour (which implies $VX < XY$). Each case contradicts the assumption that VX was the longest road in the circuit. Thus there can be no circuit in the road network.

6. Determine all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$f(n) = \begin{cases} n - 9, & \text{if } n > 2009 \\ f(f(n + 1009)), & \text{if } n \leq 2009. \end{cases}$$

Solution. Firstly, we are given

$$n > 2009 \implies f(n) = n - 9 \quad (6)$$

$$\text{and } n \leq 2009 \implies f(n) = f(f(n + 1009)). \quad (7)$$

Thus,

$$\begin{aligned} n \in [1001, 2009] &\implies n + 1009 > 2009 \\ &\implies f(n + 1009) = n + 1009 - 9, && \text{by (6)} \\ &= n + 1000 && (8) \end{aligned}$$

$$\begin{aligned} &\implies f(n) = f(f(n + 1009)), && \text{by (7), since } n \leq 2009 \\ &= f(n + 1000), && \text{by (8), since } n \geq 1001 \end{aligned} \quad (9)$$

Hence,

$$\begin{aligned} n \in [1, 1000] &\implies n + 1009 \in [1010, 2009] \subset [1001, 2009] \\ &\implies f(n) = f(f(n + 1009)), && \text{by (7), since } n \leq 2009 \\ &= f(f(n + 1009 + 1000)), && \text{by (9)} \\ &= f(f(n + 1000 + 1009)) \\ &= f(n + 1000), && \text{by (7) (backwards),} \end{aligned} \quad (10)$$

$$\therefore n \in [1, 2009] \implies f(n) = f(n + 1000), \quad \text{consolidating (9) and (10)} \quad (11)$$

Now we start to put it together:

$$\begin{aligned} n \in [2010, \infty) &\implies f(n) = n - 9, && \text{by (6)} \\ n \in [1010, 2009] &\implies f(n) = f(n + 1000), && \text{by (11), since } n \leq 2009 \\ &= n + 1000 - 9, && \text{by (6), since } n + 1000 > 2009 \\ &= n + 991 && (12) \end{aligned}$$

$$\begin{aligned} n \in [10, 1009] &\implies f(n) = f(n + 1000), && \text{by (11), since } n \leq 2009 \\ &= n + 1000 + 991, && \text{by (12), since } n + 1000 \in [1010, 2009] \\ &= n + 1991 && (13) \end{aligned}$$

$$\begin{aligned} n \in [1, 9] &\implies f(n) = f(n + 1000), && \text{by (11), since } n \leq 2009 \\ &= n + 1000 + 1991, && \text{by (13), since } n + 1000 \in [10, 2009] \\ &= n + 2991 && (14) \end{aligned}$$

Thus, by (6), (12), (13) and (14), there is just one function satisfying the given conditions, namely

$$f(n) = \begin{cases} n + 2991, & \text{if } n \in [1, 9] \\ n + 1991, & \text{if } n \in [10, 1009] \\ n + 991, & \text{if } n \in [1010, 2009] \\ n - 9, & \text{if } n > 2009. \end{cases}$$

7. Let I be the incentre of $\triangle ABC$ in which $AC \neq BC$. Let Γ be the circle passing through A , I and B . Suppose Γ intersects the line AC at A and X and intersects the line BC at B and Y .

Show that $AX = BY$.

(The *incentre* of a triangle is the intersection of its angle bisectors.)

8. Let ABC be a triangle, and let X, Y, Z be points on the sides BC, CA, AB , respectively. Let T be the area of $\triangle XYZ$, and T_1, T_2, T_3 be the areas of $\triangle AYZ, \triangle BZX, \triangle CXY$, respectively.

Prove that $\frac{3}{T} \leq \frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_3}$.

Solution. We shall use the following notation.

$(UVW) :=$ Area of $\triangle UVW$

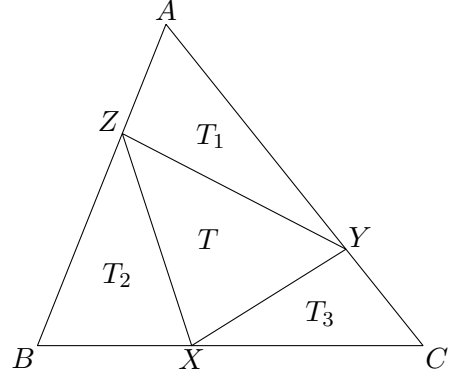
$\delta(P, \ell) :=$ (perpendicular) distance from point P to line ℓ

Then

$$\begin{aligned} \frac{T}{T_1} + 1 &= \frac{T + T_1}{T_1} \\ &= \frac{(AXY) + (AXZ)}{(AYZ)} \\ &= \frac{(AXY)}{(AYZ)} + \frac{(AXZ)}{(AYZ)} = \frac{\delta(X, AC)}{\delta(Z, AC)} + \frac{\delta(X, AB)}{\delta(Y, AB)} \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{T}{T_2} + 1 &= \frac{(BXY) + (BYZ)}{(BXZ)} = \frac{\delta(Y, BC)}{\delta(Z, BC)} + \frac{\delta(Y, AB)}{\delta(X, AB)} \\ \frac{T}{T_3} + 1 &= \frac{(CXZ) + (CYZ)}{(CXY)} = \frac{\delta(Z, BC)}{\delta(Y, BC)} + \frac{\delta(Z, AC)}{\delta(X, AC)} \end{aligned}$$



Adding the results of $T/T_i + 1, i = 1, \dots, 3$ together, we have

$$\begin{aligned} 3 + \frac{T}{T_1} + \frac{T}{T_2} + \frac{T}{T_3} &= \frac{\delta(X, AC)}{\delta(Z, AC)} + \frac{\delta(X, AB)}{\delta(Y, AB)} + \frac{\delta(Y, BC)}{\delta(Z, BC)} + \frac{\delta(Y, AB)}{\delta(X, AB)} \\ &\quad + \frac{\delta(Z, BC)}{\delta(Y, BC)} + \frac{\delta(Z, AC)}{\delta(X, AC)} \\ &= \left(\frac{\delta(X, AC)}{\delta(Z, AC)} + \frac{\delta(Z, AC)}{\delta(X, AC)} \right) + \left(\frac{\delta(X, AB)}{\delta(Y, AB)} + \frac{\delta(Y, AB)}{\delta(X, AB)} \right) \\ &\quad + \left(\frac{\delta(Y, BC)}{\delta(Z, BC)} + \frac{\delta(Z, BC)}{\delta(Y, BC)} \right) \\ &\geq 6, \quad \text{since } x + \frac{1}{x} \geq 2 \text{ for } x > 0 \end{aligned}$$

$$\begin{aligned} \therefore \frac{T}{T_1} + \frac{T}{T_2} + \frac{T}{T_3} &\geq 3 \\ \therefore \frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_3} &\geq \frac{3}{T} \\ \therefore \frac{3}{T} &\leq \frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_3} \end{aligned}$$