

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 2010 Problems

1. Let $a_1, a_2, a_3, \dots \in \mathbb{R}$ be such that

$$a_1 + a_2 = 2010$$
$$\text{and } a_{n-1}a_{n+1} = a_n \quad \text{for } n = 2, 3, 4, \dots$$

Determine all possible values of $a_{2011} + a_{2012}$.

2. A *tiling* of an $m \times m$ chessboard is a complete covering of the board, without overlap, by 2×1 tiles. (A 2×1 tile covers exactly two squares of the chessboard and may be horizontal or vertical.)

- (a) Prove that for each tiling of a 2010×4 chessboard there is a straight line dividing the board into two non-empty regions such that each tile lies completely in one of the regions.
- (b) Prove that there is a tiling of a 2010×5 chessboard such that every straight line dividing the board into two non-empty regions crosses at least one tile.

3. Consider $\triangle ABC$ with $AB \neq AC$, and P, Q, R, S are points on the line through B and C , such that P is the midpoint of BC , AQ bisects $\angle BAC$, $AR \perp BC$ and $AS \perp AQ$.

Prove that $PR \times QS = AB \times AC$.

4. A positive integer is said to be *square-free* if it is 1, is a prime, or is the product of two or more different primes.

A prime p is called *good* if for all primes $q < p$, the remainder when p is divided by q is square-free.

Determine all good primes.

5. Let K and L be concentric circles with radii r and s , respectively, where $r < s$. Let PA be a chord of K . Let BC be a chord of L passing through P , with $BC \perp PA$.

Find a formula, in terms of r and s , for $PA^2 + PB^2 + PC^2$.

6. Prove that

$$\sqrt[3]{6 + \sqrt[3]{845} + \sqrt[3]{325}} + \sqrt[3]{6 + \sqrt[3]{847} + \sqrt[3]{539}} = \sqrt[3]{4 + \sqrt[3]{245} + \sqrt[3]{175}} + \sqrt[3]{8 + \sqrt[3]{1859} + \sqrt[3]{1573}}.$$

7. Let $a, b, c, d \in \mathbb{Z}$ such that $0 < a < b < c < d < 2010$.

Prove that there exists $e \in \mathbb{Z}$ satisfying:

- (i) $0 < e < 2010$,
- (ii) $e \mid 2010$, and
- (iii) no two of a, b, c, d give the same remainder on division by e .

8. Let O be the circumcentre and H the orthocentre of $\triangle ABC$, where

$$\angle A < \angle B < \angle C < 90^\circ.$$

Prove that the incentre of $\triangle ABC$ lies inside $\triangle BHO$.