The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

### AMO TRAINING SESSIONS

# Australian Mathematics Olympiad, 2010 Problems with Some Solutions

**1.** Let $a1, a_2, a_3, \ldots \in \mathbb{R}$ be such that

$$\begin{aligned} a_1 + a_2 &= 2010 \\ \text{and } a_{n-1}a_{n+1} &= a_n \quad \text{for } n = 2, 3, 4, \ldots. \end{aligned}$$

Determine all possible values of $a_{2011} + a_{2012}$.

**Solution.** We are given that $a_n$, $n \in \mathbb{N}$ satisfies

$$a_1 + a_2 = 2010 \tag{1}$$

$$a_{n-1}a_{n+1} = a_n, n \geq 2. \tag{2}$$

Consider the following cases of the values of $a_1$ and $a_2$.

Case 1: $a_1 = 0$. Then

$$\begin{aligned} a_2 &= 2010, && \text{by (1)} \\ \text{and } a_2 &= a_1a_3 = 0, && \text{by (2), (contradiction).} \end{aligned}$$

$\therefore a_1 \neq 0$.

Case 2: $a_2 = 0$. Then

$$\begin{aligned} a_1 &= 2010, && \text{by (1)} \\ 0 = a_2 &= a_1a_3, && \text{by (2)} \\ \therefore a_3 &= 0, && \text{since } a_1 \neq 0. \end{aligned}$$

Now, $a_n = 0$ for $n \geq 3$ follows by induction, since we have $a_3 = 0$, and then for $n \geq 4$,

$$a_{n-1} = 0 \implies a_n = a_{n-1}a_{n+1} = 0, \quad \text{by (2).}$$

Thus in this case we have

$$a_{2011} + a_{2012} = 0 + 0 = 0.$$

Case 3: $a_1 \neq 0$ and $a_2 \neq 0$. Firstly, from (2) we have

$$a_{n+1} = \frac{a_n}{a_{n-1}} \text{ if } a_{n-1} \neq 0, \text{ for } n \geq 2. \tag{3}$$

Since $a_1 \neq 0$ and $a_2 \neq 0$, and (3) gives $a_{n+1} \neq 0$, if $a_{n-1} \neq 0$ and $a_n \neq 0$, we have by induction that $a_n \neq 0$ for all $n \geq 1$.

With $n = 2$, (3) gives

$$a_3 = \frac{a_2}{a_1}.$$

Using (3) on itself, we have for $n \geq 3$,

$$a_{n+1} = \frac{a_n}{a_{n-1}} = \frac{a_{n-1}}{a_{n-2}} \cdot \frac{1}{a_{n-1}} = \frac{1}{a_{n-2}}.$$

Equivalently, for $n \geq 1$,

$$a_{n+3} = \frac{1}{a_n}. \tag{4}$$

Thus we have

$$\begin{aligned} a_4 &= \frac{1}{a_1} \\ a_5 &= \frac{1}{a_2} \\ a_6 &= \frac{1}{a_3} = \frac{a_1}{a_2} \\ a_7 &= \frac{1}{a_4} = a_1 \\ a_8 &= \frac{1}{a_5} = a_2. \end{aligned}$$

Hence $a_n$ is periodic with period 6. Indeed, applying (4) to itself gives

$$a_{n+6} = a_n, \text{ for } n \geq 1.$$

Since

$$\begin{aligned} 2011 &\equiv 1 \pmod 6 \\ \text{and } 2012 &\equiv 2 \pmod 6, \end{aligned}$$

we have

$$\begin{aligned} a_{2011} + a_{2012} &= a_1 + a_2 \\ &= 2010, \quad \text{by (1)}. \end{aligned}$$

So there are two possible values of $a_{2011} + a_{2012}$, namely 0 and 2010.

**2.** A *tiling* of an $m \times m$ chessboard is a complete covering of the board, without overlap, by $2 \times 1$ tiles. (A $2 \times 1$ tile covers exactly two squares of the chessboard and may be horizontal or vertical.)

(a) Prove that for each tiling of a $2010 \times 4$ chessboard there is a straight line dividing the board into two non-empty regions such that each tile lies completely in one of the regions.

(b) Prove that there is a tiling of a $2010 \times 5$ chessboard such that every straight line dividing the board into two non-empty regions crosses at least one tile.

**3.** Consider $\triangle ABC$ with $AB \neq AC$, and $P, Q, R, S$ are points on the line through $B$ and $C$, such that $P$ is the midpoint of $BC$, $AQ$ bisects $\angle BAC$, $AR \perp BC$ and $AS \perp AQ$.

Prove that $PR \times QS = AB \times AC$.

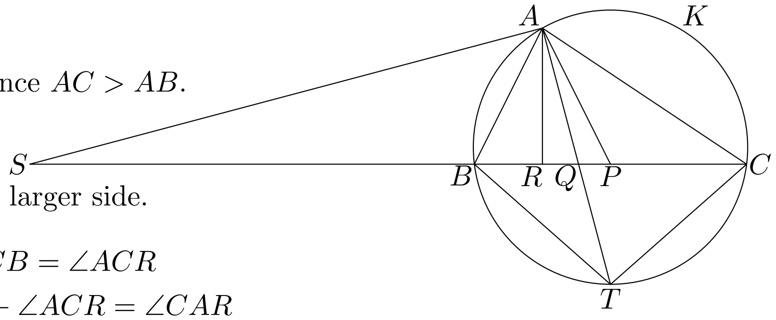
**Solution.** Without loss of generality, assume $AC > AB$. (Observe that the required result is symmetric with respect to these lengths.)

By the angle bisector theorem,

$$\frac{BQ}{QC} = \frac{AB}{AC}, \quad \text{which is} < 1 \text{ since } AC > AB.$$

So $Q$ is to the left of $P$.

Also, the larger angle is opposite the larger side.



$$\begin{aligned} \therefore \angle ABR = \angle ABC &> \angle ACB = \angle ACR \\ \therefore \angle BAR = 90^\circ - \angle ABR &< 90^\circ - \angle ACR = \angle CAR \end{aligned}$$

Since $\angle BAQ = \angle CAQ$, $R$ is to the left of $Q$.

(We note that $R$ could be on either side of $B$.)
Now, $\angle BAC < 180^\circ$. So $\angle BAQ < 90^\circ = \angle SAQ$. So, $S$ is exterior to $\triangle ABC$, and to the left of $R$. Let $K$ be the circumcircle of $\triangle ABC$, and let $AQ$ meet $K$ again at $T$. Then

$$\begin{aligned}
\angle BAT = \angle BAQ &= \angle QAC \\
\angle ATB &= \angle ACB && \text{(standing on same arc: } AB\text{)} \\
&= \angle ACQ \\
\therefore \triangle BAT &\sim \triangle QAC, && \text{by the AA Rule} \\
\therefore \frac{AB}{AQ} &= \frac{AT}{AC} \\
\therefore AB \cdot AC &= AQ \cdot AT \\
&= AQ(AQ + QT) \\
&= AQ^2 + AQ \cdot QT. && (5)
\end{aligned}$$

Since $\angle BAT = \angle QAC = \angle TAC$, the chords

$$BT = TC,$$

so that $T$ is equidistant from $B$ and $C$, and hence lies on the perpendicular bisector of $B$ and $C$, as does $P$. So $PT \perp BC (= SP)$. Hence,

$$\begin{aligned}
\angle SAQ = 90^\circ &= \angle TPQ \\
\angle AQS &= \angle PQT && \text{(vertically opposite)} \\
\therefore \triangle SAQ &\sim \triangle TPQ \\
\therefore \frac{AQ}{QS} &= \frac{PQ}{QT} \\
\therefore AQ \cdot QT &= PQ \cdot QS && (6)
\end{aligned}$$

Now

$$\begin{aligned}
\angle SAQ = 90^\circ &= \angle ARQ \\
\angle AQS &= \angle RQA && \text{(same angle)} \\
\therefore \triangle SAQ &\sim \triangle ARQ, && \text{by the AA Rule} \\
\therefore \frac{AQ}{QS} &= \frac{RQ}{QA} = \frac{QR}{AQ} \\
\therefore AQ^2 &= QR \cdot QS && (7)
\end{aligned}$$

So, finally

$$\begin{aligned}
\therefore AB \cdot AC &= AQ^2 + AQ \cdot QT, && \text{by (5)} \\
&= QR \cdot QS + PQ \cdot QS, && \text{by (7) and (6)} \\
&= (PQ + QR)QS \\
&= PR \cdot QS.
\end{aligned}$$

**Alternatively**, let $a, b, c$ be the side-lengths of $\triangle ABC$ in the standard way. Then, by the (internal) angle bisector theorem,

$$\begin{aligned}
\frac{BQ}{QC} &= \frac{c}{b} \\
\therefore BQ &= \frac{c}{b} \cdot QC \\
&= \frac{c}{b}(a - BQ) \\
b \cdot BQ &= ac - c \cdot BQ \\
(b+c)BQ &= ac \\
BQ &= \frac{ac}{b+c}.
\end{aligned}$$

Similarly, by the external bisector theorem,

$$\begin{aligned}
\frac{BS}{SC} &= \frac{c}{b} \\
\therefore BS &= \frac{c}{b} \cdot SC \\
&= \frac{c}{b}(a + BS) \\
&= \frac{ac}{b-c}, \text{ analogously to above.} \\
\therefore QS &= BQ + BS \\
&= ac\Big(\frac{1}{b+c} + \frac{1}{b-c}\Big) \\
&= \frac{2abc}{b^2 - c^2} \\
\therefore \frac{QS}{cb} = \frac{QS}{AB \cdot AC} &= \frac{2a}{b^2 - c^2}.
\end{aligned}$$

However,

$$\begin{aligned}
PR &= RC - PC \\
&= b\cos C - \frac{a}{2} \\
\text{and } c^2 &= a^2 + b^2 - 2ab\cos C \\
\therefore 2ab\cos C &= a^2 + b^2 - c^2 \\
b\cos C &= \frac{a^2 + b^2 - c^2}{2a} \\
&= \frac{a}{2} + \frac{b^2 - c^2}{2a} \\
\therefore PR &= \frac{b^2 - c^2}{2a} \\
&= \frac{AB \cdot AC}{QS} \\
\therefore AB \cdot AC &= PR \cdot QS.
\end{aligned}$$

**4.** A positive integer is said to be *square-free* if it is 1, is a prime, or is the product of two or more different primes.

A prime $p$ is called *good* if for all primes $q < p$, the remainder when $p$ is divided by $q$ is square-free.

Determine all good primes.

**5.** Let $K$ and $L$ be concentric circles with radii $r$ and $s$, respectively, where $r < s$. Let $PA$ be a chord of $K$. Let $BC$ be a chord of $L$ passing through $P$, with $BC \perp PA$.

Find a formula, in terms of $r$ and $s$, for $PA^2 + PB^2 + PC^2$.

**6.** Prove that

$$\sqrt[3]{6+\sqrt[3]{845}+\sqrt[3]{325}}+\sqrt[3]{6+\sqrt[3]{847}+\sqrt[3]{539}}=\sqrt[3]{4+\sqrt[3]{245}+\sqrt[3]{175}}+\sqrt[3]{8+\sqrt[3]{1859}+\sqrt[3]{1573}}.$$

**Solution.** First we observe that

$$\begin{aligned} 245 &= 5\cdot 7^2, & 175 &= 5^2\cdot 7, & 847 &= 7\cdot 11^2, & 539 &= 7^2\cdot 11\\ 325 &= 5^2\cdot 13, & 845 &= 5\cdot 13^2, & 1859 &= 11\cdot 13^2, & 1573 &= 11^2\cdot 13. \end{aligned}$$

Thus, for example

$$6+\sqrt[3]{845}+\sqrt[3]{325} = 6 + 5^{\frac13}\cdot(13^{\frac13})^2 + (5^{\frac13})^2\cdot 13^{\frac13}.$$

Comparing with the identity,

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

we have

$$\begin{aligned} 3\big(6 + 5^{\frac13}\cdot(13^{\frac13})^2 + (5^{\frac13})^2\cdot 13^{\frac13}\big) &= 18 + 3\cdot 5^{\frac13}\cdot(13^{\frac13})^2 + 3\cdot(5^{\frac13})^2\cdot 13^{\frac13}\\ &= 5 + 13 + 3\cdot 5^{\frac13}\cdot(13^{\frac13})^2 + 3\cdot(5^{\frac13})^2\cdot 13^{\frac13}\\ &= (5^{\frac13} + 13^{\frac13})^3\\ \therefore\ \sqrt[3]{6+\sqrt[3]{845}+\sqrt[3]{325}} &= \frac{\sqrt[3]{5}+\sqrt[3]{13}}{\sqrt[3]{3}}. \end{aligned}$$

Similarly,

$$\begin{aligned} \sqrt[3]{6+\sqrt[3]{847}+\sqrt[3]{539}} &= \frac{\sqrt[3]{7}+\sqrt[3]{11}}{\sqrt[3]{3}}, && \text{since } 3\cdot 6 = 18 = 7 + 11\\ \sqrt[3]{4+\sqrt[3]{245}+\sqrt[3]{175}} &= \frac{\sqrt[3]{5}+\sqrt[3]{7}}{\sqrt[3]{3}}, && \text{since } 3\cdot 4 = 12 = 5 + 7\\ \sqrt[3]{8+\sqrt[3]{1859}+\sqrt[3]{1573}} &= \frac{\sqrt[3]{11}+\sqrt[3]{13}}{\sqrt[3]{3}}, && \text{since } 3\cdot 8 = 24 = 11 + 13. \end{aligned}$$

So, finally we have

$$\begin{aligned} \sqrt[3]{6+\sqrt[3]{845}+\sqrt[3]{325}}+\sqrt[3]{6+\sqrt[3]{847}+\sqrt[3]{539}} &= \frac{\sqrt[3]{5}+\sqrt[3]{13}}{\sqrt[3]{3}} + \frac{\sqrt[3]{7}+\sqrt[3]{11}}{\sqrt[3]{3}}\\ &= \frac{\sqrt[3]{5}+\sqrt[3]{7}}{\sqrt[3]{3}} + \frac{\sqrt[3]{11}+\sqrt[3]{13}}{\sqrt[3]{3}}\\ &= \sqrt[3]{4+\sqrt[3]{245}+\sqrt[3]{175}}+\sqrt[3]{8+\sqrt[3]{1859}+\sqrt[3]{1573}}. \end{aligned}$$

**7.** Let $a, b, c, d \in \mathbb{Z}$ such that $0 < a < b < c < d < 2010$.

Prove that there exists $e \in \mathbb{Z}$ satisfying:

(i) $0 < e < 2010$,

(ii) $e \,|\, 2010$, and

(iii) no two of $a, b, c, d$ give the same remainder on division by $e$.

**8.** Let $O$ be the circumcentre and $H$ the orthocentre of $\triangle ABC$, where

$$\angle A < \angle B < \angle C < 90^\circ.$$

Prove that the incentre of $\triangle ABC$ lies inside $\triangle BHO$.