

The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS
AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 2011 Problems

1. Let $2 \leq b \in \mathbb{N}$. For $n \in \mathbb{N}$ define $S_b(n)$ to be the sum of the digits of n when n is expressed in base b , e.g. if $b = 4$ then

$$S_4(26) = S_4(1 \times 4^2 + 2 \times 4 + 2 \times 1) = 1 + 2 + 2 = 5.$$

Determine all $m \in \mathbb{N}$ with the property that

for all $n \in \mathbb{N}$, whenever m is a factor of $S_b(n)$, then m is also a factor of $S_b(n+1) - 1$.

2. The vertices of the regular polygon $P_1P_2 \dots P_{2n}$ lie on a circle of radius 1. For $1 \leq i \leq n-1$, let a_i be the length of the line segment P_1P_{i+1} .

Prove that

$$\left(\frac{4}{a_1^2} - 1\right)\left(\frac{4}{a_2^2} - 1\right) \dots \left(\frac{4}{a_{n-1}^2} - 1\right) = 1.$$

3. Let A, B, C be three distinct points on a circle of radius r .

Prove that $\triangle ABC$ has an obtuse angle if and only if there exists a point X in the plane such that the distances AX, BX and CX are all less than r .

4. Determine all functions $f : \mathbb{N} \cup \{0\} \rightarrow \mathbb{N} \cup \{0\}$ such that $f(1) > 0$ and

$$(f(x))^2 + (f(y))^2 = f(x^2 + y^2), \forall x, y \in \mathbb{N} \cup \{0\}.$$

5. Let $PQRS$ be a rectangle with centre O . Let E, F be the midpoints of PQ, QR , respectively. Let A, B, C, D be points on PQ, QR, RS, SP , respectively, such that $ABCD$ is a rhombus and A lies between E and Q . Let K be the intersection of AB and EF .

Prove that $OK \perp AB$.

6. Determine all $r \in \mathbb{R}$ such that the three solutions of the equation

$$x^3 - 30x^2 + rx - 780 = 0$$

are the side lengths of a right-angled triangle.

7. Determine all $n \in \mathbb{N}$ with the property:

for all $k \in \mathbb{Z}$, there exists $i \in \mathbb{Z}$ such that $\frac{i(i-1)}{2} - k$ is divisible by n .

8. Let S be the set of all positive integers less than or equal to 2211. Let T be a subset of S containing 2011 elements.

Prove that there is a number in T that is the sum of 11, not necessarily distinct numbers from T .