The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 2011 Problems with Some Solutions

1. Let $2 \le b \in \mathbb{N}$. For $n \in \mathbb{N}$ define $S_b(n)$ to be the sum of the digits of n when n is expressed in base b, e.g. if $b = 4$ then

$$S_4(26) = S_4(1 \times 4^2 + 2 \times 4 + 2 \times 1) = 1 + 2 + 2 = 5.$$

Determine all $m \in \mathbb{N}$ with the property that

for all $n \in \mathbb{N}$, whenever m is a factor of $S_b(n)$, then m is also a factor of $S_b(n+1) - 1$.

Solution. Consider an $n \in \mathbb{N}$. Let its digit representation in base b be $(a_k a_{k-1} \dots a_0)_b$, except that for convenience we choose $a_k = 0$, so that the index ℓ of the first of these digits that is not equal to $b-1$, is always defined. (If $a_0 \ne b-1$ then $\ell = 0$, and, if $a_{k-1} = a_{k-2} = \dots = a_1 = a_0 = b-1$ then $\ell = k$.) We need to consider when:

$$m \,|\, S_b(n) \implies m \,|\, S_b(n+1) - 1,$$

or equivalently, when:

$$S_b(n) \equiv 0 \pmod{m} \implies S_b(n+1) - 1 \equiv 0 \pmod{m} \tag{1}$$

$$\begin{aligned} n &= (a_k a_{k-1} \dots a_0)_b \\ &= a_k \cdot b^k + \dots + a_\ell \cdot b^\ell + (b-1)\cdot b^{\ell-1} + \dots + (b-1)\cdot 1, \qquad \text{by how we defined } \ell \\ &= \sum_{i=\ell}^{k} a_i b^i + \sum_{i=0}^{\ell-1} (b-1) b^i \\ \therefore S_b(n) &= a_k + \dots + a_\ell + (b-1)\ell \\ &= \sum_{i=\ell}^{k} a_i + (b-1)\ell. \end{aligned}$$

Similarly,

$$n + 1 = a_k \cdot b^k + \dots + a_{\ell+1} \cdot b^{\ell+1} + (a_\ell + 1) \cdot b^\ell + 0 \cdot b^{\ell-1} + \dots + 0 \cdot 1$$

$$\therefore S_b(n+1) - 1 = \sum_{i=\ell}^{k} a_i.$$

In general, given $A \equiv 0 \pmod{m}$, to deduce $B \equiv 0 \pmod{m}$, it is necessary that $B - A \equiv 0 \pmod{m}$. Thus, from (1) it follows that we require, for our given n,

$$(b-1)\ell \equiv 0 \pmod{m}, \tag{2}$$

where we note that the choice of n determines ℓ, and as n ranges over $\mathbb{N}$, ℓ ranges over $\mathbb{N} \cup \{0\}$. Thus to determine all m satisfying the given problem is equivalent to determining all m such that (2) holds for all $\ell \in \mathbb{N} \cup \{0\}$. Now,

$$(b-1)\ell \equiv 0 \pmod{m}\ \forall \ell \in \mathbb{N} \cup \{0\} \implies (b-1) \equiv 0 \pmod{m}, \quad \text{considering } \ell = 1.$$

Also,

$$(b-1) \equiv 0 \pmod{m} \implies (b-1)\ell \equiv 0 \pmod{m}\ \forall \ell \in \mathbb{N} \cup \{0\}.$$

Thus, the required $m \in \mathbb{N}$ are precisely those that satisfy:

$$(b-1) \equiv 0 \pmod{m},$$

i.e. those m that are positive divisors of $b - 1$.

Alternative Method. As noted in the first solution, the problem is to find those $m \in \mathbb{N}$ for which, for any $n \in \mathbb{N}$,

$$S_b(n) \equiv 0 \pmod{m} \implies S_b(n+1) - 1 \equiv 0 \pmod{m} \tag{1}$$

We start by proving a result that generalises a familiar result (where $b = 10$).

Lemma. *If* $m \mid b-1$ *then* $S_b(n) \equiv n \pmod{m}$.

Proof. Let the digit representation of n in base b be $(a_k a_{k-1} \dots a_0)_b$, and assume $m \mid b-1$. Then

$$\begin{aligned}
b - 1 &\equiv 0 \pmod{m}\\
b &\equiv 1 \pmod{m}\\
\therefore n &= (a_k a_{k-1} \dots a_0)_b\\
&= \sum_{i=0}^{k} a_i b^i\\
&\equiv \sum_{i=0}^{k} a_i 1^i \pmod{m}\\
&= \sum_{i=0}^{k} a_i = S_b(n)\\
\therefore n &\equiv S_b(n) \pmod{m}.
\end{aligned}$$

□

Now assume $m \mid b-1$. Then, by the lemma,

$$\begin{aligned}
S_b(n+1) &\equiv n+1 \pmod{m}\\
\therefore S_b(n+1) - 1 &\equiv n \pmod{m}\\
&\equiv S_b(n) \pmod{m}
\end{aligned}$$

and, so, in particular, (1) is satisfied whenever $m \mid b-1$.
Next assume m is such that (1) is satisfied (for all $n \in \mathbb{N}$). Consider $n = (A0A \dots A)_b$ of $m+1$ digits, where A represents the digit $b-1$. Then $n+1 = (A10\dots0)_b$ and

$$S_b(n) = m(b-1) \equiv 0 \pmod{m},$$

and so,

$$\begin{aligned}
S_b(n+1) &= b - 1 + 1 = b\\
\therefore \text{ by (1), } b - 1 &= S_b(n+1) - 1\\
&\equiv 0 \pmod{m}
\end{aligned}$$

i.e. if m satisfies (1) for all $n \in \mathbb{N}$, then $m \mid b-1$.
Thus we have shown (1) is satisfied for all $n \in \mathbb{N}$, if and only if $m \mid b-1$.

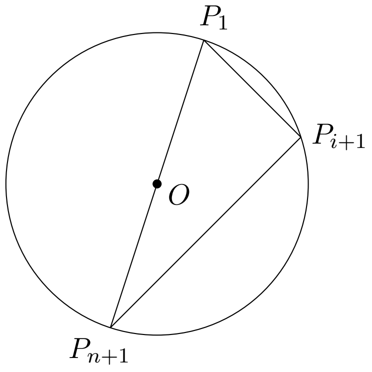
2. The vertices of the regular polygon $P_1P_2\dots P_{2n}$ lie on a circle of radius 1. For $1 \le i \le n-1$, let a_i be the length of the line segment P_1P_{i+1}.

Prove that

$$\left(\frac{4}{a_1^2} - 1\right)\left(\frac{4}{a_2^2} - 1\right)\dots\left(\frac{4}{a_{n-1}^2} - 1\right) = 1.$$

Solution. Observe that P_1P_{n+1} is a diameter of the circle. Also, by symmetry,

$$P_{i+1}P_{n+1} = P_1P_{n-i+1} = a_{n-i}.$$



Thus $\triangle P_1P_{i+1}P_{n+1}$ is a triangle in a semicircle, with sides $a_i, a_{n-i}, 2$, and hence, by Pythagoras' Theorem,

$$\begin{aligned}
a_i^2 + a_{n-i}^2 &= 4\\
\therefore 4 - a_i^2 &= a_{n-i}^2.
\end{aligned}$$

$$\begin{aligned}
\therefore \Big(\frac{4}{a_1^2} - 1\Big)\Big(\frac{4}{a_2^2} - 1\Big)\dots\Big(\frac{4}{a_{n-1}^2} - 1\Big) &= \prod_{i=1}^{n-1}\Big(\frac{4}{a_i^2} - 1\Big)\\
&= \prod_{i=1}^{n-1}\Big(\frac{4 - a_i^2}{a_i^2}\Big)\\
&= \prod_{i=1}^{n-1}\Big(\frac{a_{n-i}^2}{a_i^2}\Big)\\
&= \frac{\prod_{i=1}^{n-1} a_{n-i}^2}{\prod_{i=1}^{n-1} a_i^2}\\
&= \frac{\prod_{i=1}^{n-1} a_i^2}{\prod_{i=1}^{n-1} a_i^2}\\
&= 1.
\end{aligned}$$

3. Let A, B, C be three distinct points on a circle of radius r.

Prove that $\triangle ABC$ has an obtuse angle if and only if there exists a point X in the plane such that the distances AX, BX and CX are all less than r.

4. Determine all functions $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ such that $f(1) > 0$ and

$$\big(f(x)\big)^2 + \big(f(y)\big)^2 = f(x^2 + y^2),\ \forall x, y \in \mathbb{N} \cup \{0\}.$$

Solution. First observe that $f(x) = x$ reduces

$$\big(f(x)\big)^2 + \big(f(y)\big)^2 = f(x^2 + y^2) \tag{3}$$

to the identity

$$x^2 + y^2 = x^2 + y^2$$

and, in this case, $f(1) = 1 > 0$. Hence, $f(x) = x$ is a solution. We will show it is the only solution. Putting $x = y = 0$ in (3) gives

$$\begin{aligned}
\big(f(0)\big)^2 + \big(f(0)\big)^2 &= f(0)\\
2\big(f(0)\big)^2 - f(0) &= 0\\
f(0)\big(2f(0) - 1\big) &= 0\\
\therefore f(0) &= 0, && \text{since } f(0) \in \mathbb{N} \cup \{0\}.
\end{aligned}$$

Putting $x = 1, y = 0$ in (3) gives

$$\begin{aligned} \big(f(1)\big)^2 + \big(f(0)\big)^2 &= f(1) \\ \big(f(1)\big)^2 - f(1) &= 0, &&\text{since } f(0) = 0 \\ f(1)\big(f(1) - 1\big) &= 0 \\ \therefore f(1) &= 1, &&\text{since } f(1) > 0 \text{ (given).} \end{aligned}$$

Putting $x = y = 1$ in (3) gives

$$\begin{aligned} \big(f(1)\big)^2 + \big(f(1)\big)^2 &= f(2) \\ \therefore f(2) &= 0, &&\text{since } f(1) = 1. \end{aligned}$$

Now letting $x = 2$ and in turn putting $y = 0$, 1 and then 2, we have:

$$\begin{aligned} (x,y) = (2,0) &\implies f(4) = f(2^2 + 0^2) = f(2)^2 + f(0)^2 = 4 \\ (x,y) = (2,1) &\implies f(5) = f(2^2 + 1^2) = f(2)^2 + f(1)^2 = 5 \\ (x,y) = (2,0) &\implies f(8) = f(2^2 + 2^2) = f(2)^2 + f(2)^2 = 8. \end{aligned}$$

Thus, so far, we have $f(x) = x$ for $x = 0, 1, 2, 4, 5, 8$.
The following lemma and ensuing corollary are key to the remaining argument.

Lemma. *Given* (3),

$$t^2 + u^2 = v^2 + w^2 \implies \big(f(t)\big)^2 + \big(f(u)\big)^2 = \big(f(v)\big)^2 + \big(f(w)\big)^2.$$

Proof. Assume $t^2 + u^2 = v^2 + w^2$. Then

$$\begin{aligned} \big(f(t)\big)^2 + \big(f(u)\big)^2 &= t^2 + u^2, &&\text{by (3)} \\ &= v^2 + w^2 \\ &= \big(f(v)\big)^2 + \big(f(w)\big)^2, &&\text{by (3).} \end{aligned}$$

□

Corollary. *If f is a solution of the given problem, and $t, u, v, w \in \mathbb{N} \cup \{0\}$ are such that*

$$t^2 + u^2 = v^2 + w^2,\ f(u) = u,\ f(v) = v \text{ and } f(w) = w$$

then $f(t) = t$.

Proof. Since $t^2 + u^2 = v^2 + w^2$, we have

$$t^2 = v^2 + w^2 - u^2.$$

By the lemma, we have

$$\begin{aligned} \big(f(t)\big)^2 + \big(f(u)\big)^2 &= \big(f(v)\big)^2 + \big(f(w)\big)^2 \\ \therefore \big(f(t)\big)^2 &= \big(f(v)\big)^2 + \big(f(w)\big)^2 - \big(f(u)\big)^2 \\ &= v^2 + w^2 - u^2, &&\text{since } f(u) = u, f(v) = v, f(w) = w \\ &= t^2 \\ \therefore f(t) &= t, &&\text{since } f(t) \geq 0. \end{aligned}$$

□

Since $3^2 + 4^2 = 0^2 + 5^2$ and we know $f(x) = x$ for $x = 0, 4, 5$, by the corollary we have $f(3) = 3$. Now letting $x = 3$ and in turn putting $y = 0$ and then 1, we have:

$$\begin{aligned}
(x, y) = (3, 0) &\implies \quad f(9) = f(3^2 + 0^2) = f(3)^2 + f(0)^2 = 9 \\
(x, y) = (2, 1) &\implies f(10) = f(3^2 + 1^2) = f(3)^2 + f(1)^2 = 10.
\end{aligned}$$

Thus, by the corollary,

$$\begin{aligned}
f(6) &= 6, &&\text{since } 6^2 + 8^2 = 0^2 + 10^2 \text{ and } f(x) = x \text{ for } x = 0, 8, 10 \\
f(7) &= 7, &&\text{since } 7^1 + 1^2 = 5^2 + 5^2 \text{ and } f(x) = x \text{ for } x = 1, 5.
\end{aligned}$$

So now we have $f(x) = x$ for $x = 0, 1, 2, \dots, 10$.
We complete the argument by an induction that needs the following two identities:

$$(2m+1)^2 + (m-2)^2 = (2m-1)^2 + (m+2)^2 \tag{4}$$

$$(2m+2)^2 + (m-4)^2 = (2m-2)^2 + (m+4)^2 \tag{5}$$

Proof of identities (4) and (5). Consider,

$$\begin{aligned}
(2m+a)^2 - (2m-a)^2 &= \big((2m+a) + (2m-a)\big)\big((2m+a) - (2m-a)\big) \\
&= (4m) \cdot 2a \\
&= (2m) \cdot 4a \\
&= \big((m+2a) + (m-2a)\big)\big((m+2a) - (m-2a)\big) \\
&= (m+2a)^2 - (m-2a)^2 \\
\therefore (2m+a)^2 + (m-2a)^2 &= (2m-a)^2 + (m+2a)^2.
\end{aligned}$$

Now the identities (4) and (5) are the cases where $a = 1$ and $a = 2$, respectively. $\square$

Notice that for $a = 1, 2$ we have

$$2m + a > 2m - a > m - 2a$$

and the condition $2m + a > m + 2a$ is equivalent to $m > a$.
Now we are ready to prove, by induction, the claim that, under the given conditions of f,

$$P(n) : f(n) = n$$

for all $n \in \mathbb{N} \cup \{0\}$.
We have $P(n)$ holds for $n = 0, 1, 2, \dots, 10$.
Now, we show for $k > 10$, that $P(0), P(1), \dots, P(k-1) \implies P(k)$.
Assume $k > 10$ and $P(0), P(1), \dots, P(k-1)$. We have two cases:

Case 1: k is odd, say $k = 2m + 1$. Then $k > 10$ implies $m > 4 > 1$. Thus we have $2m + 1$ is greater than each of $(m - 2)$, $(2m - 1)$ and $(m + 2)$ and hence, by the inductive assumption, $f(x) = x$ for $x = m - 2, 2m - 1, m + 2$, so that with the identity (4)

$$(2m+1)^2 + (m-2)^2 = (2m-1)^2 + (m+2)^2$$

we have

$$f(2m+1) = 2m + 1,$$

by the corollary.

Case 2: k is even, say $k = 2m + 2$. Then $k > 10$ implies $m > 4 > 2$. Thus we have $2m + 2$ is greater than each of $(m - 4)$, $(2m - 2)$ and $(m + 4)$ and hence, by the inductive assumption, $f(x) = x$ for $x = m - 4, 2m - 2, m + 4$, so that with the identity (5)

$$(2m + 2)^2 + (m - 4)^2 = (2m - 2)^2 + (m + 4)^2$$

we have

$$f(2m + 2) = 2m + 2,$$

by the corollary.

Thus in either case $P(k) : f(k) = k$ follows.

Hence, the induction is complete, and we have shown that

$$f(n) = n \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Thus $f(x) = x$ is the only solution to the given problem as claimed.

5. Let $PQRS$ be a rectangle with centre O. Let E, F be the midpoints of PQ, QR, respectively. Let A, B, C, D be points on PQ, QR, RS, SP, respectively, such that $ABCD$ is a rhombus and A lies between E and Q. Let K be the intersection of AB and EF.

Prove that $OK \perp AB$.

6. Determine all $r \in \mathbb{R}$ such that the three solutions of the equation

$$x^3 - 30x^2 + rx - 780 = 0$$

are the side lengths of a right-angled triangle.

7. Determine all $n \in \mathbb{N}$ with the property:

for all $k \in \mathbb{Z}$, there exists $i \in \mathbb{Z}$ such that $\dfrac{i(i-1)}{2} - k$ is divisible by n.

8. Let S be the set of all positive integers less than or equal to 2211. Let T be a subset of S containing 2011 elements.

Prove that there is a number in T that is the sum of 11, not necessarily distinct numbers from T.

Solution. We start by defining 201 disjoint subsets of S:

$\{201, 2211 = 11 \cdot 201\}$,
$\{200, 2200 = 11 \cdot 200\}$,
$\vdots$
$\{20, 220 = 11 \cdot 20\}$,
$\{19, 209 = 11 \cdot 19\}$,

$\{18, 221, 401 = 10 \cdot 18 + 221\}$,
$\{17, 219, 2207 = 17 + 10 \cdot 219\}$,
$\{16, 218, 2196 = 16 + 10 \cdot 218\}$,
$\vdots$
$\{8, 210, 2108 = 8 + 10 \cdot 210\}$,

$\{7, 208, 2087 = 7 + 10 \cdot 208\}$,
$\{6, 207, 2076 = 6 + 10 \cdot 207\}$,
$\vdots$
$\{1, 202, 2021 = 1 + 10 \cdot 202\}$.

Number the subsets by their first element. Then these subsets are disjoint since,

their first elements are the integers $1, 2, \dots, 201$;
subsets $1, 2, \dots, 20$ have as their second elements (in a slightly adjusted order),
the integers $202, 203, \dots, 221$;
subsets $21, 22, \dots, 36$ have as their second elements,
the integers $231 = 11 \cdot 21, 242 = 11 \cdot 22, \dots, 396 = 11 \cdot 36$;
subset 18 has third element 401;

subsets $37, 38, \dots, 183$ have as their second elements,
the integers $407 = 11 \cdot 37, 418 = 11 \cdot 38, \dots, 2013 = 11 \cdot 183$;
subsets $1, 2, \dots, 7$ have as their third elements $2021, 2032, \dots, 2087$,
all > 2013 and $\equiv 8 \pmod{11}$;
subsets $8, 9, \dots, 17$ have as their third elements $2108, 2119, \dots, 2207$,
all > 2087 and $\equiv 7 \pmod{11}$; and
subsets $183, 184, \dots, 201$ have as their second elements, the integers
$2024 = 11 \cdot 184, 2035 = 11 \cdot 185, \dots, 2211 = 11 \cdot 201$, all > 2013 and $\equiv 0 \pmod{11}$.

Also, the elements all lie between 1 and 2211, inclusive, and hence each subset is a subset of S.
Observe that each subset is of form $\{t_1, \dots, t_\ell, t\}$ where

$$t = \sum_{i=1}^{\ell} \alpha_i t_i, \text{ with each } \alpha_i \in \mathbb{N} \text{ and } \sum_{i=1}^{\ell} \alpha_i = 11.$$

Thus, if we can show that one of the 201 subsets is a subset of T, then we will have shown there is a $t \in T$ that is the sum of 11, not necessarily distinct numbers (namely, the corresponding $t_1, \dots, t_\ell$) from T.
Suppose for a contradiction, that none of the 201 subsets of S given above is a subset of T. Then at least one member of each of these subsets is not in T, i.e. at least 201 elements of S are not in T, so that

$$\begin{aligned} |T| &\le |S| - 201 \\ &= 2211 - 201 = 2010 \end{aligned}$$

which is a contradiction, since $|T| = 2011$ (given).
Thus, there is a number in T that is the sum of 11, not necessarily distinct numbers from T.