

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 2012 Problems

1. Determine the largest $n \in \mathbb{N}$ such that

$$4^n + 2^{2012} + 1$$

is a perfect square.

2. Let S be a set of points in the (x, y) -plane such that

- (i) no three points in S are collinear, and
- (ii) for any two points (a, b) and (c, d) in S , at least two of a, b, c, d are equal.

What is the largest possible number of points in S ?

3. Let P be the intersections of the diagonals of a convex quadrilateral $ABCD$. Let X, Y, Z be points on sides AB, BC, CD , respectively, such that

$$AX : XB = BY : YC = CZ : ZD = 2.$$

Suppose further that XY is tangent to the circumcircle of $\triangle CYZ$ and YZ is tangent to the circumcircle of $\triangle BXY$.

Prove that $\angle APD = \angle XYZ$.

4. Let

$$p(x) = x^{2012} + a_{2011}x^{2011} + a_{2010}x^{2010} + \cdots + a_2x^2 + a_1x$$

be a polynomial such that

$$p(x_1) = p(x_2) = \cdots = p(x_{2012}) = 0$$

for distinct integers $x_1, x_2, \dots, x_{2012}$.

Determine all $n \in \mathbb{Z}$ such that $p(p(n)) = 0$.

5. (a) Prove that there is a unique triple (x, y, z) of *real* numbers, satisfying all three equations

$$(x + 20)(y + 12)(z - 4) = 1215$$

$$(x + 20)(y - 12)(z + 4) = 1215$$

$$(x - 20)(y + 12)(z + 4) = 1215.$$

- (b) Determine the least integer $n > 1215$ such that the system

$$(x + 20)(y + 12)(z - 4) = n$$

$$(x + 20)(y - 12)(z + 4) = n$$

$$(x - 20)(y + 12)(z + 4) = n,$$

has *integer* solutions x, y, z .

6. Determine all triples (p, a, b) of positive integers, such that p is prime, $a \leq b$ and $p^a + p^b$ is a perfect cube.

7. A rectangular array of square tiles has at least 3 rows and at least 3 columns.

Each tile is coloured red, blue or green.

The bottom row consists entirely of red tiles, and in the top row all tiles are green. Except for the top and bottom tiles, all tiles in the leftmost column are blue, and except for the bottom tile, all tiles in the rightmost column are green.

Prove that there is a point where 4 tiles meet, at which there is at least one tile of each colour.

8. Two circles C_1 and C_2 intersect at distinct points A and B . Let P be the point on C_1 and Q the point on C_2 such that PQ is the common tangent closer to B than to A .

Let BQ intersect C_1 again at R , and let BP intersect C_2 again at S . Let M be the midpoint of PR and let N be the midpoint of QS .

Prove that AB bisects $\angle MAN$.