The University of Western Australia
DEPARTMENT OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

Australian Mathematics Olympiad, 2013 Problems

1. Let $ABCD$ be a parallelogram, and let P be a point on side CD. Let the line through P parallel to AD intersect the diagonal AC at Q.

Prove
$$(BCP)^2 = (QBP) \cdot (ABP).$$

2. Determine all triples (x, y, z) of positive integers that satisfy
$$20x^x = 13y^y + 7z^z.$$

3. Let $n \in \mathbb{N}$ such that $8 \mid n - 3$. Let $3 < k \in \mathbb{Z}$.

Prove that 2^k divides $n^{2^{k-3}} - 2^{k-1} - 1$.

4. Let $S = \{1, 2, 3, \ldots, 2013\}$. A subset of S is said to be *good* if among its elements there are numbers $a_1 < a_2 < \cdots < a_{300}$ such that
$$a_i - i \quad \text{is a multiple of 3 for} \quad 1 \le i \le 300.$$
Find the smallest integer n such that every subset of S with n elements is good.

5. Find the largest $n \in \mathbb{N}$ such that 2013 can be written as the sum of the squares of n different positive integers.

6. There are 2013 people at a party. Among any 3 of these people the number of pairs of people who know each other is odd.

Prove that there are 1007 people who all know each other.

7. Let $\triangle ABC$ be acute-angled with $\angle ABC = 35°$. Let $I = \text{incentre}(ABC)$ and that $AI + AC = BC$. Let P be the point such that PB and PC are tangent to circumcircle(ABC). Let Q be the point on the line AB such that $PQ \parallel AC$.

Find the value of $\angle AQC$.

8. Find all $n \in \mathbb{N}$ for which there exist $x_1, x_2, \ldots, x_n \in \mathbb{R}$ such that

(i) $-1 < x_i < 1$ for $i = 1, 2, \ldots, n$,
(ii) $x_1 + x_2 + \cdots + x_n = 0$ and
(iii) $\sqrt{1 - x_1^2} + \sqrt{1 - x_2^2} + \cdots + \sqrt{1 - x_n^2} = 1.$