DAY 1
Tuesday, 11 February 2014
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 0$ and, for $n \geq 2$,

$$a_n = \max_{i=1,\ldots,n-1} \left\{ i + a_i + a_{n-i} \right\}.$$

(For example, $a_2 = 1$ and $a_3 = 3$.)

Determine a_{200}.

2. Let ABC be a triangle with $\angle BAC < 90^\circ$. Let k be the circle through A that is tangent to BC at C. Let M be the midpoint of BC, and let AM intersect k a second time at D. Finally, let BD (extended) intersect k a second time at E.

Prove that $\angle BAC = \angle CAE$.

3. Consider labelling the twenty vertices of a regular dodecahedron with twenty different integers. Each edge of the dodecahedron can then be labelled with the number $|a - b|$, where a and b are the labels of its endpoints. Let e be the largest edge label.

What is the smallest possible value of e over all such vertex labellings?

(A regular dodecahedron is a polyhedron with twelve identical regular pentagonal faces.)

4. Let $\mathbb{N}^+$ denote the set of positive integers, and let $\mathbb{R}$ denote the set of real numbers.

Find all functions $f : \mathbb{N}^+ \to \mathbb{R}$ that satisfy the following three conditions:

(i) $f(1) = 1$,

(ii) $f(n) = 0$ if n contains the digit 2 in its decimal representation,

(iii) $f(mn) = f(m)f(n)$ for all positive integers m, n.

DAY 2

Wednesday, 12 February 2014
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. Determine all non-integer real numbers x such that

$$x + \frac{2014}{x} = \lfloor x \rfloor + \frac{2014}{\lfloor x \rfloor}.$$

(Note that $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to the real number x. For example, $\lfloor 20.14 \rfloor = 20$ and $\lfloor -20.14 \rfloor = -21$.)

6. Let S be the set of all numbers

$$a_0 + 10\,a_1 + 10^2\,a_2 + \cdots + 10^n\,a_n \qquad (n = 0, 1, 2, \ldots)$$

where

(i) a_i is an integer satisfying $0 \le a_i \le 9$ for $i = 0, 1, \ldots, n$ and $a_n \neq 0$, and

(ii) $a_i < \dfrac{a_{i-1} + a_{i+1}}{2}$ for $i = 1, 2, \ldots, n-1$.

Determine the largest number in the set S.

7. Let ABC be a triangle. Let P and Q be points on the sides AB and AC, respectively, such that BC and PQ are parallel. Let D be a point inside triangle APQ. Let E and F be the intersections of PQ with BD and CD, respectively. Finally, let O_E and O_F be the circumcentres of triangle DEQ and triangle DFP, respectively.

Prove that $O_E O_F$ is perpendicular to AD.

8. An $n \times n$ square is tiled with 1×1 tiles, some of which are coloured. Sally is allowed to colour in any uncoloured tile that shares edges with at least three coloured tiles. She discovers that by repeating this process all tiles will eventually be coloured.

Show that initially there must have been more than $\frac{n^2}{3}$ coloured tiles.