DAY 1
Tuesday, 11 February 2014
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 0$ and, for $n \geq 2$,

$$a_n = \max_{i=1,\ldots,n-1} \left\{ i + a_i + a_{n-i} \right\}.$$

(For example, $a_2 = 1$ and $a_3 = 3$.)

Determine a_{200}.

2. Let ABC be a triangle with $\angle BAC < 90°$. Let k be the circle through A that is tangent to BC at C. Let M be the midpoint of BC, and let AM intersect k a second time at D. Finally, let BD (extended) intersect k a second time at E.

Prove that $\angle BAC = \angle CAE$.

3. Consider labelling the twenty vertices of a regular dodecahedron with twenty different integers. Each edge of the dodecahedron can then be labelled with the number $|a - b|$, where a and b are the labels of its endpoints. Let e be the largest edge label.

What is the smallest possible value of e over all such vertex labellings?

(A regular dodecahedron is a polyhedron with twelve identical regular pentagonal faces.)

4. Let $\mathbb{N}^+$ denote the set of positive integers, and let $\mathbb{R}$ denote the set of real numbers.

Find all functions $f : \mathbb{N}^+ \to \mathbb{R}$ that satisfy the following three conditions:

(i) $f(1) = 1$,

(ii) $f(n) = 0$ if n contains the digit 2 in its decimal representation,

(iii) $f(mn) = f(m)f(n)$ for all positive integers m, n.

DAY 2
Wednesday, 12 February 2014
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. Determine all non-integer real numbers x such that

$$x + \frac{2014}{x} = \lfloor x \rfloor + \frac{2014}{\lfloor x \rfloor}.$$

(Note that $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to the real number x. For example, $\lfloor 20.14 \rfloor = 20$ and $\lfloor -20.14 \rfloor = -21$.)

6. Let S be the set of all numbers

$$a_0 + 10\,a_1 + 10^2\,a_2 + \cdots + 10^n\,a_n \qquad (n = 0, 1, 2, \ldots)$$

where

(i) a_i is an integer satisfying $0 \le a_i \le 9$ for $i = 0, 1, \ldots, n$ and $a_n \ne 0$, and

(ii) $a_i < \dfrac{a_{i-1} + a_{i+1}}{2}$ for $i = 1, 2, \ldots, n-1$.

Determine the largest number in the set S.

7. Let ABC be a triangle. Let P and Q be points on the sides AB and AC, respectively, such that BC and PQ are parallel. Let D be a point inside triangle APQ. Let E and F be the intersections of PQ with BD and CD, respectively. Finally, let O_E and O_F be the circumcentres of triangle DEQ and triangle DFP, respectively.

Prove that $O_E O_F$ is perpendicular to AD.

8. An $n \times n$ square is tiled with 1×1 tiles, some of which are coloured. Sally is allowed to colour in any uncoloured tile that shares edges with at least three coloured tiles. She discovers that by repeating this process all tiles will eventually be coloured.

Show that initially there must have been more than $\frac{n^2}{3}$ coloured tiles.

1. **Solution 1** (Leo Li, year 10, Christ Church Grammar School, WA)

Answer: 19900.

We prove $a_n = 0 + 1 + \cdots + (n-1)$ by strong induction.

The base case $n = 1$ is given.

For the inductive part, assume $a_n = 0 + 1 + \cdots + (n-1)$ for $n \le k$. We know

$$a_{k+1} = \max_{i=1,\ldots,k} \{i + a_i + a_{k+1-i}\}. \tag{1}$$

We claim that the max in (1) occurs when $i = k$. For this it is sufficient to prove that whenever $i < k$, we have

$$k + a_k + a_1 > i + a_i + a_{k+1-i}.$$

But since $a_1 = 0$ and $k > i$, it suffices to prove that

$$a_k - a_i \ge a_{k+1-i}. \tag{2}$$

Using the inductive assumption, we have

$$\begin{aligned} a_k - a_i &= (0 + 1 + \cdots + (k-1)) - (0 + 1 + \cdots + (i-1)) \\ &= i + (i+1) + \cdots + (k-1) \end{aligned} \tag{3}$$

and

$$\begin{aligned} a_{k+1-i} &= 0 + 1 + \cdots + (k + 1 - i - 1) \\ &= 1 + 2 \cdots + (k-i). \end{aligned} \tag{4}$$

Observe that the right hand sides of (3) and (4) both consist of the sum of $(n-k)$ consecutive integers and that the first term in (3) is at least as big as the first term in (4). This establishes the truth of (2), and hence also our claim.

Therefore, using our claim, we may substitute $i = k$ in (1) to find

$$a_{k+1} = k + 0 + 1 + \cdots + (k-1) + 0 = 0 + 1 + \cdots + k.$$

This completes the induction.

Finally, the well-known formula for summing the first so many consecutive positive integers may be used to deduce

$$a_{200} = 1 + 2 + \cdots + 200 = \frac{199 \times 200}{2} = 19900,$$

as required. $\square$

Solution 2 (Kevin Xian, year 10, James Ruse Agricultural High School, NSW)

We prove $a_n = \frac{n(n-1)}{2}$ by strong induction.

The base case $a_1 = 0$ is already given.

For the inductive part, assume $a_n = \frac{n(n-1)}{2}$ for $n \leq k$. For $n = k+1$ we have

$$\begin{aligned}
a_{k+1} &= \max_{i=1,\ldots,k} \{i + a_i + a_{k+1-i}\} \\
&= \max_{i=1,\ldots,k} \left\{ i + \frac{i(i-1)}{2} + \frac{(k+1-i)(k-i)}{2} \right\} \\
&= \max_{i=1,\ldots,k} \left\{ i^2 - ki + \frac{k^2+k}{2} \right\} \\
&= \frac{k^2+k}{2} + \max_{i=1,\ldots,k} \{i(i-k)\}.
\end{aligned}$$

However, $i(i-k) < 0$ for $1 \leq i \leq k-1$, while $i(i-k) = 0$ for $i = k$. Therefore,

$$\max_{i=1,\ldots,k} \{i(i-k)\} = 0,$$

and so,

$$\begin{aligned}
a_{k+1} &= \frac{k^2+k}{2} \\
&= \frac{(k+1)((k+1)-1)}{2}.
\end{aligned}$$

This completes the induction. Hence in particular,

$$\begin{aligned}
a_{200} &= \frac{200 \times 199}{2} \\
&= 19900,
\end{aligned}$$

as desired. $\square$

Solution 3 (Jerry Mao, year 8, Caulfield Grammar School, VIC)

We prove $a_n = \frac{n(n-1)}{2}$ by strong induction.

The base case $n = 1$ is given.

For the inductive part, assume $a_n = \frac{n(n-1)}{2}$ for $n \le k$. We know

$$a_{k+1} = \max_{i=1,\ldots,k} \{i + a_i + a_{k+1-i}\}. \tag{1}$$

We claim that the max in (1) occurs when $i = k$.

Assume, for the sake of contradiction, that the max occurs for some integer $i = r$ satisfying $1 \le r \le k - 1$.

Case 1. $r < \frac{k}{2}$.

Consider $s = k + 1 - i$. Note that $\frac{k}{2} < s \le k$. Furthermore,

$$\begin{aligned} r + a_r + a_{k+1-r} &< s + a_{k+1-r} + a_r \\ &= s + a_s + a_{k+1-s}, \end{aligned}$$

in contradiction to the assumption that the max in (1) occurs at $i = r$.

Case 2. $\frac{k}{2} \le r \le k - 1$.

We claim that $r + a_r + a_{k+1-r} < r + 1 + a_{r+1} + a_{k-(r+1)}$.

Using the inductive assumption we know

$$\begin{aligned} r + a_r + a_{k+1-r} &= r + \frac{r(r-1)}{2} + \frac{(k+1-r)(k-r)}{2} \\ &= r^2 - rk + \frac{k^2}{2} + \frac{k}{2} \end{aligned}$$

and

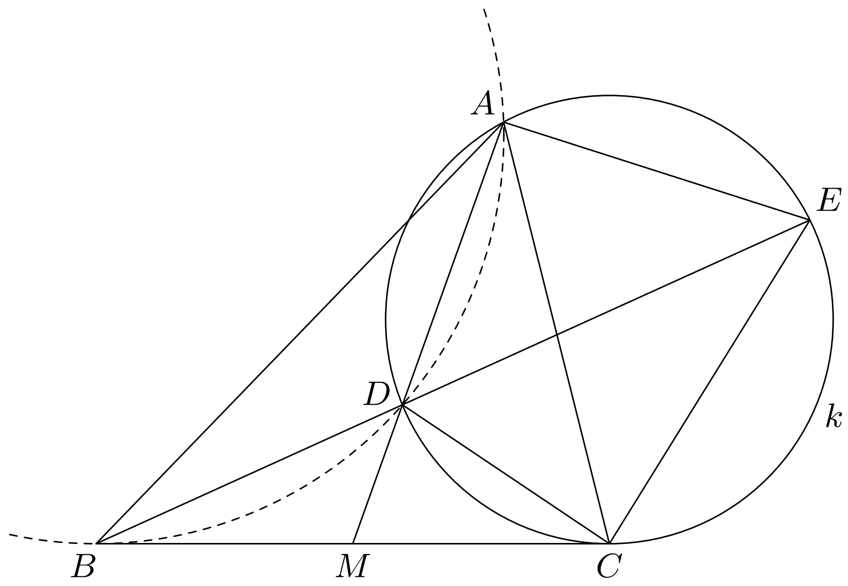
$$\begin{aligned} r + 1 + a_{r+1} + a_{k-(r+1)} &= r + 1 + \frac{(r+1)r}{2} + \frac{(k-r)(k-r-1)}{2} \\ &= r^2 - rk + \frac{k^2}{2} + \frac{k}{2} + 2r + 1 - k. \end{aligned}$$

So our claim is true because $k \le 2r$. Since the claim is true we have contradicted the assumption that the max in (1) occurs at $i = r$.

Since cases 1 and 2 both end up in contradictions, the max must occur at $i = k$. Substituting $i = k$ in (1), using the inductive assumption and simplifying yields $a_{k+1} = \frac{(k+1)k}{2}$. This completes the induction.

Thus we may conclude $a_{200} = \frac{200 \times 199}{2} = 19900$, as required. $\square$

2. **Solution 1** (Seyoon Ragavan, year 10, Knox Grammar School, NSW)



Since MC is tangent to circle k at C, then by the power of a point theorem we have

$$MC^2 = MD \cdot MA.$$

Since $MB = MC$ it follows that

$$MB^2 = MD \cdot MA.$$

Hence considering the power of M with respect to circle ADB, it follows that MB is tangent to circle ADB at M.

In the angle chase that follows, AST is an abbreviation for the alternate segment theorem.

$$\begin{aligned}
\angle BAC &= \angle BAM + \angle MAC \\
&= \angle MBD + \angle DAC && \text{(AST circle } ADB\text{)} \\
&= \angle CBD + \angle BCD && \text{(AST circle } k\text{)} \\
&= \angle CDE && \text{(exterior angle } \triangle BCD\text{)} \\
&= \angle CAE, && (AECD \text{ cyclic})
\end{aligned}$$

which is the desired result. □

Solution 2 (Hannah Sheng, year 10, Rossmoyne Senior High School, WA)

Refer to the diagram used in solution 1.

Since BC is tangent to circle k at C, we may apply the alternate segment theorem to deduce

$$\angle MAC = \angle DAC = \angle MCD. \tag{1}$$

Furthermore, since $\angle CMA = \angle CMD$, it follows by (AA) that

$$\triangle MCD \sim \triangle MAC.$$

Therefore,

$$\frac{MC}{MA} = \frac{MD}{MC}.$$

Since $MC = MB$, we have

$$\frac{MB}{MA} = \frac{MD}{MB}.$$

But now $\angle BMA = \angle BMD$, and so by (PAP) we have

$$\triangle MBD \sim \triangle MAB.$$

Hence

$$\angle BAM = \angle MBA = \angle CBA. \tag{2}$$

Finally, adding (1) and (2) together yields

$$\begin{aligned}\angle BAC &= \angle MCD + \angle MBD \\ &= \angle CDE && \text{(exterior angle } \triangle BCD) \\ &= \angle CAE, && (AECD \text{ cyclic})\end{aligned}$$

as required. $\square$

Comment Solutions 1 and 2 are essentially the same solution. This is because the similar triangles used in solution 2 are exactly the same similar triangles that are normally used to prove the power of a point theorem that was used in solution 1.

Solution 3 (Yang Song, year 11, James Ruse Agricultural High School, NSW)

Refer to the diagram used in solution 1.

As in solution 2, we deduce

$$\triangle MBD \sim \triangle MAB.$$

It follows that

$$\begin{aligned}\angle ABC &= \angle ABM \\ &= \angle BDM \quad (\triangle MBD \sim \triangle MAB) \\ &= \angle ADE \\ &= \angle ACE. \quad (AECD \text{ cyclic})\end{aligned}$$

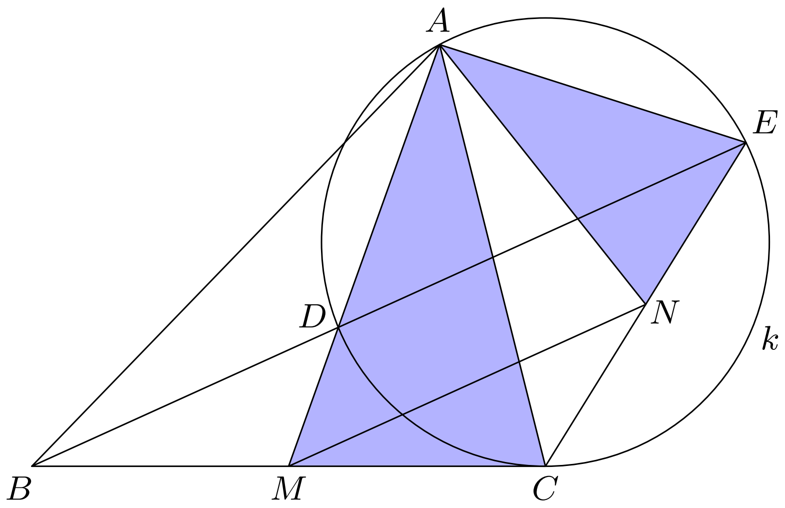
Since also $\angle ACE = \angle AEC$ from the alternate segment theorem applied to circle k, it follows by (AA) that

$$\triangle ABC \sim \triangle ACE.$$

From this we may immediately conclude that $\angle BAC = \angle CAE$. $\square$

Solution 4 (Matthew Sun, year 12, Penleigh and Essendon Grammar School, VIC)

Let N be the midpoint of CE.



Since M is the midpoint of BC and N is the midpoint of EC, it follows that $MN \parallel BE$. Using this along with the fact that $AECD$ is cyclic, we find

$$\angle CNM = \angle CEB = \angle CED = \angle CAD = \angle CAM,$$

from which it follows that $CMAN$ is cyclic.

Hence, $\angle CMA = \angle ENA$. By the alternate segment theorem we have $\angle ACM = \angle AEC = \angle AEN$. So by (AA) we have $\triangle ACM \sim \triangle AEN$.

One implication of this is

$$\angle MAC = \angle NAE. \tag{3}$$

Another implication is

$$\frac{AM}{AN} = \frac{MC}{NE} = \frac{MB}{NC},$$

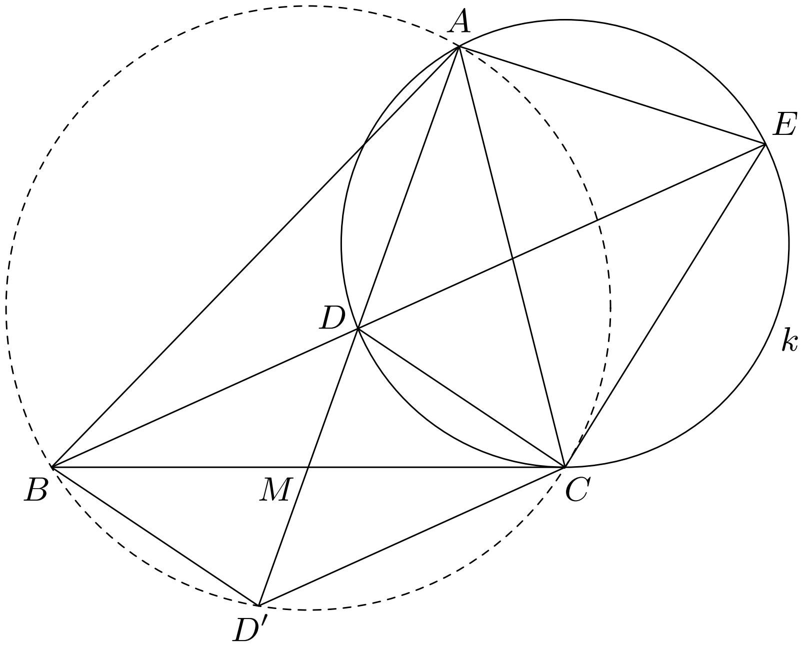
since $MC = MB$ and $NE = NC$. And we have $\angle AMB = \angle ANC$ due to $CMAN$ being cyclic. Thus by (PAP) we have $\triangle AMB \sim \triangle ANC$ and so

$$\angle BAM = \angle CAN. \tag{4}$$

Adding together (3) and (4) yields the required result. □

Solution 5 (Richard Gong, year 9, Sydney Grammar School, NSW)

Let D' be the point so that $BDCD'$ is a parallelogram. Since the diagonals of a parallelogram bisect each other and M is the midpoint of BC, it follows that M is the midpoint of DD'. Therefore, D' is collinear with D and M.



It follows that

$$\begin{aligned}\angle BD'A &= \angle D'DC \quad (BD' \parallel DC)\\ &= \angle AEC \quad (AECD \text{ cyclic})\\ &= \angle BCA, \quad (\text{alternate segment theorem})\end{aligned}$$

and so $ABD'C$ is cyclic.

Therefore, $\angle ABC = \angle AD'C = \angle DD'C$. Part of the above angle chase yielded $\angle D'DC = \angle BCA$. Hence $\triangle BAC \sim \triangle D'CD$ by (AA).

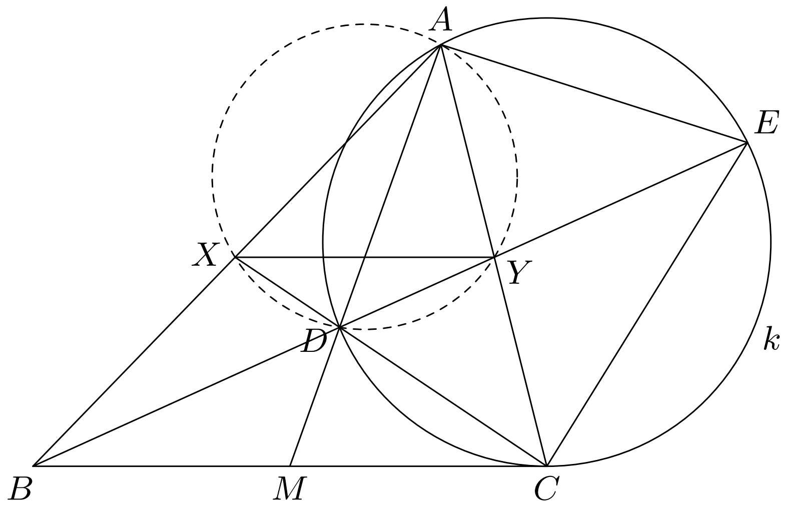
Therefore,

$$\begin{aligned}\angle BAC &= \angle D'CD\\ &= \angle CDE \quad (D'C \parallel BD)\\ &= \angle CAE, \quad (AECD \text{ cyclic})\end{aligned}$$

as required. $\square$

Solution 6 (Allen Lu, year 11, Sydney Grammar School, NSW)

Let lines CD and AB intersect at point X, and let lines BE and AC intersect at point Y.



Since AM, BY and CX we may apply Ceva's theorem to find

$$\frac{BM}{MC} \cdot \frac{CY}{YA} \cdot \frac{AX}{XB} = 1$$

$$\Rightarrow \quad \frac{AX}{XB} = \frac{AY}{YC} \quad \text{(since } BM = MC\text{)}.$$

It follows that $XY \parallel BC$.

Therefore,

$$\begin{aligned} \angle DXY &= \angle DCB \quad (XY \parallel BC) \\ &= \angle DAC, \quad \text{(alternate segment theorem)} \end{aligned}$$

from which it follows that $AXDY$ is cyclic.

Therefore,

$$\begin{aligned} \angle BAC &= \angle EDC \quad (AXYD \text{ cyclic}) \\ &= \angle CAE, \quad (AECD \text{ cyclic}) \end{aligned}$$

as desired. $\square$

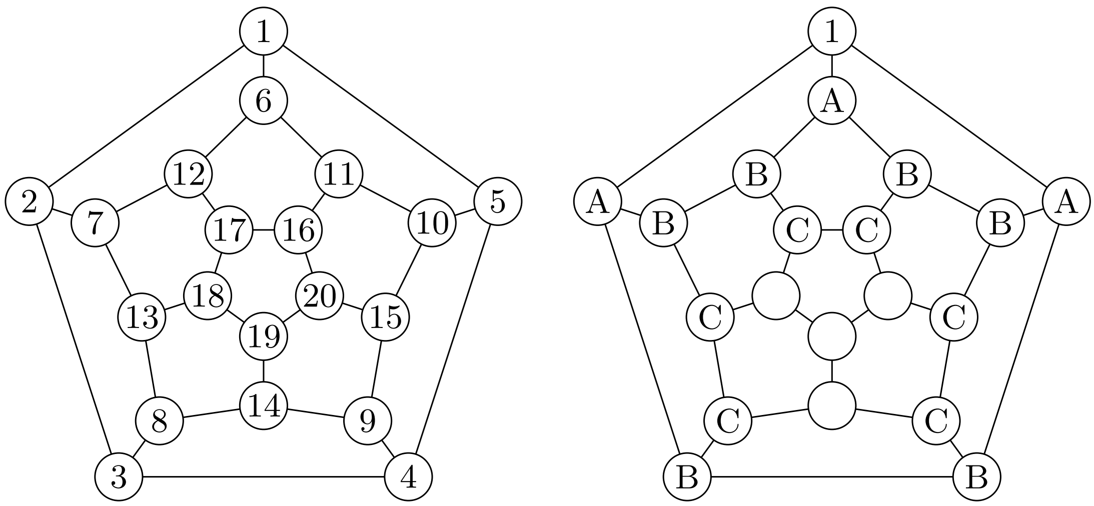
Comment Solutions 5 and 6 are quite similar underneath the surface. See if you can find the connection!

3. **Solution** (Jerry Mao, year 8, Caulfield Grammar School, VIC)

Answer: $e = 6$.

Since the edge labels only depend on the difference of the vertex labels we can assume without loss of generality that the vertex with minimal label is labelled with the number 1. The first diagram below shows a graph of the dodecahedron along with a numbering for which $e = 6$.

In the second diagram below, the vertices marked with A, B and C, require at least 1, 2 and 3 edges, respectively, to reach them from the vertex labelled 1.



If we assume $e \leq 5$, then each A-vertex has label at most 6, each B-vertex has label at most 11 and each C-vertex has label at most 16. Thus all of the 16 vertices including the one labelled with 1 and the 15 marked with A, B or C, must be labelled with different positive integers less than or equal to 16. Therefore, they have exactly the labels $1, 2, \ldots, 16$ in some order.

Consequently, the four unmarked vertices have labels at least equal to 17. But all six C-vertices are adjacent to one of these unmarked vertices. Since $e \leq 5$, the labels of the C-vertices must all be at least $17 - e \geq 12$. Thus the six C-vertices have labels lying in the range 12 to 16. This is clearly impossible by the pigeonhole principle because all the labels are different. Hence $e \geq 6$. $\square$

4. **Solution 1** (Michael Cherryh, year 11, Gungahlin College, ACT)

Answer: $f(n) = 0$ for all integers $n \geq 2$.

Suppose $n = 2k$ is even. Then $f(n) = f(2)f(k) = 0$.

Our strategy will be to prove that for each odd integer $n > 1$, there exists a positive integer k such that n^k begins with a 2. Then since $f(n)^k = f(n^k) = 0$ it will follow that $f(n) = 0$.

By the pigeonhole principle there exist two integers $a > b$ such that n^a and n^b have the same first two digits, which we shall denote by x and y. Let us write n^a and n^b in scientific notation. That is,

$$n^a = x.ya_2a_3\ldots \times 10^k \quad \text{and} \quad n^b = x.yb_2b_3\ldots \times 10^\ell,$$

for some non-negative integers $k \geq \ell$. Then

$$n^{a-b} = \frac{x.ya_2a_3\ldots}{x.yb_2b_3\ldots} \times 10^{k-\ell} = r \times 10^{k-\ell}.$$

Note that $r \neq 1$ because n is odd and $a > b$.

If we can find a positive integer k such that r^k starts with a 2 when written in scientific notation, then we will be done because $n^{k(a-b)}$ will also start with a 2.

Our idea is as follows. If $r > 1$, then the sequence $r, r^2, r^3, \ldots$ grows arbitrarily large. But since r is close to 1 we cannot jump from being less than 2 to at least 3. Thus there is a power of r that lies between 2 and 3. Similarly, if $r < 1$, then the sequence $r, r^2, r^3, \ldots$ converges to 0. But since r is close to 1 we cannot jump from being at least 0.3 to less than 0.2 and so there is a power of r lying between 0.2 and 0.3.

If $r > 1$, then

$$1 < r < \frac{x.y + 0.1}{x.y} = 1 + \frac{0.1}{x.y} \leq 1 + 0.1 = 1.1$$

Consider the least positive integer k such that $r^k \geq 2$. Then we have $r^{k-1} < 2$. Thus $r^k < 2 \times 1.1 = 2.2$, and so r^k starts with a 2.

If $r < 1$, then

$$1 > r > \frac{x.y}{x.y + 0.1} = 1 - \frac{0.1}{x.y + 0.1} \geq 1 - \frac{0.1}{1.1} > 0.9.$$

Consider the least positive integer k such that $r^k < 0.3$. Then we have $r^{k-1} \geq 0.3$. Thus $r^k > 0.3 \times 0.9 = 0.27$, and so r^k starts with a 2.

In both cases we have shown that the required k exists. $\square$

Solution 2 (Mel Shu, year 12, Melbourne Grammar School, VIC)

We shall prove that $f(n) = 0$ for all integers $n \geq 2$. Since f is completely multiplicative it suffices to show that $f(p) = 0$ for all primes p. Since $f(p^k) = f(p)^k$ for any positive integer k it is enough to prove that any prime has a power that contains the digit 2. In fact we shall prove that any prime has a power whose first digit is 2.

Let p be any prime. We seek integers $i > 0$ and $j \geq 0$ such that

$$\begin{aligned} 2 \cdot 10^j \leq \quad & p^i \quad < 3 \cdot 10^j \\ \Leftrightarrow \quad j + \log 2 \leq & \, i \log p < j + \log 3, \end{aligned}$$

where the logarithm is to base 10. It would be a good idea to estimate the sizes of $\log 2$ and $\log 3$. Indeed since $2^9 < 10^3$ and $3^9 > 10^4$ we have $\log 2 < \frac{3}{9}$ and $\log 3 > \frac{4}{9}$. Hence it suffices to find i and j satisfying

$$\begin{aligned} & \frac{3}{9} < i\alpha - j < \frac{4}{9} \\ \Leftrightarrow \quad & \frac{3}{9} < \{i\alpha\} < \frac{4}{9}, \end{aligned} \tag{*}$$

where $\alpha = \log p$, and $\{i\alpha\}$ denotes the fractional part of $i\alpha$.

We claim that α, and consequently also $\{i\alpha\}$, are irrational. Indeed, if $\alpha = \frac{a}{b}$ for $a, b \in \mathbb{N}^+$, then $p^b = 10^a$. But then p^b would be divisible by both 2 and 5. This is impossible and so $\alpha \notin \mathbb{Q}$ as claimed.

Consider the ten irrational numbers $\{\alpha\}, \{2\alpha\}, \ldots, \{10\alpha\}$ and the nine open intervals $\left(0, \frac{1}{9}\right)$, $\left(\frac{1}{9}, \frac{2}{9}\right)$, $\ldots$, $\left(\frac{8}{9}, 1\right)$. By the pigeonhole principle at least one of these intervals contains at least two of the ten values. Suppose that $\{k\alpha\}$ and $\{\ell\alpha\}$, where $1 \leq \ell < k \leq 10$, both lie within one of these nine intervals. Let $\beta = \{k\alpha\} - \{\ell\alpha\}$. Note that $|\beta| < \frac{1}{9}$ and that $\beta \neq 0$, because $\{(k - \ell)\alpha\}$ is irrational.

Case 1: $\beta > 0$.
By adding β to itself enough times, we see that there is a positive integer m such that $\frac{3}{9} < m\beta < \frac{4}{9}$. This is because we cannot jump the interval $\left(\frac{3}{9}, \frac{4}{9}\right)$ just by adding β.

Case 2: $\beta < 0$.
By adding β to itself enough times, we see that there is a positive integer m such that $\frac{3}{9} - 1 < m\beta < \frac{4}{9} - 1$.

In both cases 1 and 2, we can satisfy (*) by taking $i = (k - \ell)m$. This concludes the proof. $\square$

Solution 3 (Angelo Di Pasquale, AMOC Senior Problems Committee)

As in solution 2, it is sufficient to prove that any prime p has a power that contains the digit 2.

We verify directly that $2 = 2^1$ and $5^2 = 25$.

Consider any other prime p. It has last digit 1, 3, 7 or 9. Since $1^4 \equiv 3^4 \equiv 7^4 \equiv 9^4 \equiv 1 \pmod{10}$, we have $p^4 = 10x + 1$ for some positive integer x. It follows that $p^8 = 100x^2 + 20x + 1$. Hence the last two digits of p^8 are 01, 21, 41, 61 or 81.

One can easily check that $41 \to 81 \to 61 \to 21 \pmod{100}$ upon repeated squaring. This shows that if p^8 does not end in 01, then p has a power that contains the digit 2 in its second last position.

Lemma. Let $m > 1$ be any integer that ends in 01. Let $k \geq 3$ be the integer such that the last k digits of m are $x0\ldots01$ where $x \neq 0$ and there are $k-2$ zeros. If $x \neq 5$, then there exists a power of m that contains a 2.

Proof. The last k digits of m^2 are $y0\ldots01$ where $y \equiv 2x \pmod{10}$. Since $x \neq 0$ or 5, it follows that y is a nonzero even digit. One may check that the last k digits go as

$$40\ldots01 \to 80\ldots01 \to 60\ldots01 \to 20\ldots01,$$

upon repeated squaring. This establishes that some power of m will contain the digit 2 in the kth last position. $\square$

The lemma solves the problem unless the last k digits of p^8 are $50\ldots01$. In such a case let w be the next digit to the left of the digit 5. Thus the last $k+1$ digits of p^8 are $w50\ldots01$.

If $w = 2$, we are finished.

If $w \neq 2$, we square again and note that the last $k+1$ digits of p^{16} are $z0\ldots01$ where $z \equiv 2w + 1 \pmod{10}$. Note that $z \neq 0$. If $z \neq 5$, we may use the lemma to solve the problem. If $z = 5$, this corresponds to $w = 2$ or 7. Since $w \neq 2$ we have $w = 7$.

We are now left with the case of when the last $k+1$ digits of p^8 are $750\ldots01$. Since $k \geq 3$ there is at least one zero among the last $k+1$ digits. Cubing $750\ldots01$, we find that the last $k+1$ digits of p^{24} are $250\ldots01$. This has a 2 in the $(k+1)$th place from the right and therefore solves the problem. $\square$

Solution 4 (Alexander Gunning, year 11, Glen Waverley Secondary College, VIC)

Clearly $f(2) = 0$. Also $f(5)^2 = f(25) = 0$, which implies $f(5) = 0$. Thus if n is any positive integer that is divisible by 2 or 5, then $f(n) = 0$. From here on we assume that n is not divisible by 2 or 5.

By Fermat's little theorem we have $5 \mid n^4 - 1$. Let $5^m \parallel n^4 - 1$.[1]

Lemma 1. If $5^m \parallel n^4 - 1$, then also $5^m \parallel n^{4a} - 1$ for $a \in \mathbb{N}^+$ and $5 \nmid a$.

Proof. Write $n^4 = 5^m k + 1$ where $5 \nmid k$. Then

$$n^{4a} - 1 = (5^m k + 1)^a - 1 = \sum_{i=1}^{a} \binom{a}{i} (5^m k)^i \equiv 5^m ka \pmod{5^{m+1}}.$$

Since $5^m ka$ is divisible by 5^m but not 5^{m+1}, the lemma is proven. □

Lemma 2. For any positive integer b we have $n^{2^m b} \equiv 1 \pmod{2^{m+1}}$.

Proof. An application of Euler's theorem yields $n^{2^m} \equiv 1 \pmod{2^{m+1}}$. The lemma follows once we raise both sides to the power of b. □

Let $d = \max\{2^m, 4\} = \operatorname{lcm}\{2^m, 4\}$. Then the two lemmas tell us that $n^d \equiv 1 \pmod{2^{m+1}}$ and $5^m \parallel n^d - 1$. Thus we may write

$$n^d = 5^m 2^{m+1} c + 1 = 2c \cdot 10^m + 1,$$

where $5 \nmid c$. Then for any integer $e \geq 2$ we have

$$\begin{aligned} n^{de} &= (2c \cdot 10^m + 1)^e \\ &= 1 + \binom{e}{1} 2c \cdot 10^m + \binom{e}{2} (2c \cdot 10^m)^2 + \cdots \\ &\equiv 2ce \cdot 10^m + 1 \pmod{10^{m+1}}. \end{aligned}$$

Since $\gcd(c, 5) = 1$, we may choose e so that $ce \equiv 1 \pmod 5$. This implies that $2ce \equiv 2 \pmod{10}$ and so we have

$$n^{de} \equiv 2 \cdot 10^m + 1 \pmod{10^{m+1}}.$$

This contains the digit 2 in the $(m+1)$th place from the right. Thus $f(n)^{de} = f(n^{de}) = 0$ and so $f(n) = 0$. □

[1] For a prime p and an integer k, the notation $p^m \parallel k$ means that $p^m \mid k$ but $p^{m+1} \nmid k$.

5. **Solution** (Kevin Xian, year 10, James Ruse Agricultural High School, NSW)

Answer: $x = -\frac{2014}{45}$.

The given equation may be rewritten as

$$\begin{aligned} x - \lfloor x \rfloor &= 2014\left(\frac{1}{\lfloor x \rfloor} - \frac{1}{x}\right) \\ &= \frac{2014(x - \lfloor x \rfloor)}{x\lfloor x \rfloor}. \end{aligned}$$

Since x is not an integer it follows that $x \neq \lfloor x \rfloor$. Hence we may divide both sides by $x - \lfloor x \rfloor$ and rearrange to find

$$x\lfloor x \rfloor = 2014. \tag{1}$$

Case 1. $\lfloor x \rfloor \geq 45$.

Then $x > 45$, and so $x\lfloor x \rfloor > 45^2 = 2025 > 2014$.

Case 2. $-44 \leq \lfloor x \rfloor \leq 44$.

Then $-44 < x < 45$, and so $x\lfloor x \rfloor < 44 \times 45 = 1980 < 2014$.

Case 3. $\lfloor x \rfloor \leq -46$.

Then $x < -45$, and so $x\lfloor x \rfloor > 45 \times 46 = 2070 > 2014$.

Case 4. $\lfloor x \rfloor = -45$.

Then from (1) we derive $x = -\frac{2014}{45} = -44\frac{34}{45}$.

Checking this in the original equation we have

$$\text{LHS} = x + \frac{2014}{x} = -\frac{2014}{45} + \frac{2014}{-\frac{2014}{45}} = -\frac{2014}{45} - 45$$

and

$$\text{RHS} = \lfloor x \rfloor + \frac{2014}{\lfloor x \rfloor} = -45 + \frac{2014}{-45} = \text{LHS},$$

as required. $\square$

6. **Solution 1** (Seyoon Ragavan, year 10, Knox Grammar School, NSW)

Answer: 96433469.

It is straightforward to verify that 96433469 is in S. Assume that S contains a number $N > 96433469$. If N has nine or more digits, let the first nine such digits in order from the left be $a, b, c, d, e, f, g, h, i$. Condition (ii) implies that

$$\begin{aligned} & 2b < a + c \\ \Rightarrow \quad & a > 2b - c \\ \Rightarrow \quad & a \geq 2b - c + 1, \end{aligned} \tag{1}$$

since a, b, c are all integers. Similarly, we deduce the following.

$$b \geq 2c - d + 1 \tag{2}$$
$$c \geq 2d - e + 1 \tag{3}$$
$$d \geq 2e - f + 1 \tag{4}$$
$$e \geq 2f - g + 1 \tag{5}$$
$$f \geq 2g - h + 1 \tag{6}$$
$$g \geq 2h - i + 1 \tag{7}$$

Since $a \leq 9$, we may use successive substitution to find the following.

$$\begin{aligned}
& a \geq 2b - c + 1 && \text{(using (1))} \\
\Rightarrow \quad & 8 \geq 2b - c && \text{(1')} \\
& \quad \geq 2(2c - d + 1) - c + 1 && \text{(using (2))} \\
\Rightarrow \quad & 6 \geq 3c - 2d && \text{(2')} \\
& \quad \geq 3(2d - e + 1) - 2d && \text{(using (3))} \\
\Rightarrow \quad & 3 \geq 4d - 3e && \text{(3')} \\
& \quad \geq 4(2e - f + 1) - 3e && \text{(using (4))} \\
\Rightarrow \quad & -1 \geq 5e - 4f && \text{(4')} \\
& \quad \geq 5(2f - g + 1) - 4f && \text{(using (5))} \\
\Rightarrow \quad & -6 \geq 6f - 5g && \text{(5')} \\
& \quad \geq 6(2g - h + 1) - 5g && \text{(using (6))} \\
\Rightarrow \quad & -12 \geq 7g - 6h && \text{(6')} \\
& \quad \geq 7(2h - i + 1) - 6h && \text{(using (7))} \\
\Rightarrow \quad & -19 \geq 8h - 7i && \text{(7')}
\end{aligned}$$

Concentrating on (4') we have $-1 \geq 5e - 4f \geq -4f$. Hence $f \geq 1$.

Substituting successively into (5'), (6') and (7') yields

$$\begin{aligned}
-6 \geq 6f - 5g \geq 6 - 5g \quad &\Rightarrow \quad g \geq 3 \\
-12 \leq 7g - 6h \geq 21 - 6h \quad &\Rightarrow \quad h \geq 6 \\
-19 \leq 8h - 7i \geq 48 - 7i \quad &\Rightarrow \quad i \geq 10.
\end{aligned}$$

However, this is in contradiction with i being a digit. Therefore, N cannot contain more than eight digits.

We are left to deal with the case where N is an eight-digit number. Let the digits of N in order from the left be a, b, c, d, e, f, g, h. We deduce inequalities (1)–(6) and (1')–(6') on the previous page in the same way as we did earlier.

Since h is a digit we know that $h \leq 9$. If we substitute successively into (6'), (5'), (4'), (3'), (2') and (1'), we find the following.

$$\begin{array}{rclcl}
-12 \geq 7g - 6h & \Rightarrow & 7g \leq 6h - 12 \leq 6 \times 9 - 12 = 42 & \Rightarrow & g \leq 6 \\
-6 \geq 6f - 5g & \Rightarrow & 6f \leq 5g - 6 \leq 5 \times 6 - 6 = 24 & \Rightarrow & f \leq 4 \\
-1 \geq 5e - 4f & \Rightarrow & 5e \leq 4f - 1 \leq 4 \times 4 - 1 = 15 & \Rightarrow & e \leq 3 \\
3 \geq 4d - 3e & \Rightarrow & 4d \leq 3e + 3 \leq 3 \times 3 + 3 = 12 & \Rightarrow & d \leq 3 \\
6 \geq 3c - 2d & \Rightarrow & 3c \leq 2d + 6 \leq 2 \times 3 + 6 = 12 & \Rightarrow & c \leq 4 \\
8 \geq 2b - c & \Rightarrow & 2b \leq c + 8 \leq 4 + 8 = 12 & \Rightarrow & b \leq 6
\end{array}$$

We also know that $a \leq 9$ because a is a digit.

Therefore, each digit of N is less than or equal to the corresponding digit of 96433469. It follows that $N \leq 96433469$. This contradicts that $N > 96433469$. Hence 96433460 is the largest number in S. $\square$

Solution 2 (Found independently by Norman Do and Ivan Guo, AMOC Senior Problems Committee)

Consider the differences $b_i = a_{i+1} - a_i$ for $i = 0, 1, 2, \ldots$. Condition (ii) is equivalent to $b_0, b_1, b_2, \ldots$ being a strictly increasing sequence.

Lemma. At most three b_i are strictly positive and at most three b_i are strictly negative.

Proof. Suppose there are four b_i that are strictly positive. If b_s is the smallest such b_i, then we have $b_s \geq 1$, $b_{s+1} \geq 2$, $b_{s+2} \geq 3$ and $b_{s+3} \geq 4$. Therefore,

$$\begin{aligned} a_{s+4} - a_s &= b_s + b_{s+1} + b_{s+2} + b_{s+3} \\ &\geq 1 + 2 + 3 + 4 \\ &= 10. \end{aligned}$$

However, this is a contradiction because a_{r+4} and a_r are single digits and hence differ by at most 9. A similar argument shows that no four b_i are strictly negative. □

It follows from the lemma that $n \leq 7$. If $n = 7$, then we must have

$$b_0 < b_1 < b_2 < 0, \quad b_3 = 0 \quad \text{and} \quad 0 < b_4 < b_5 < b_6.$$

Since the b_i are distinct integers this implies that $b_0 \leq -3$, $b_1 \leq -2$ and $b_2 \leq -1$. Hence we have the following.

$$\begin{aligned} a_0 &\leq 9 \\ a_1 &= a_0 + b_0 \leq 9 - 3 = 6 \\ a_2 &= a_1 + b_1 \leq 6 - 2 = 4 \\ a_3 &= a_2 + b_2 \leq 4 - 1 = 3 \end{aligned}$$

Similarly, coming from the other end we have $b_6 \geq 3$, $b_5 \geq 2$ and $b_4 \geq 1$. Hence we have following.

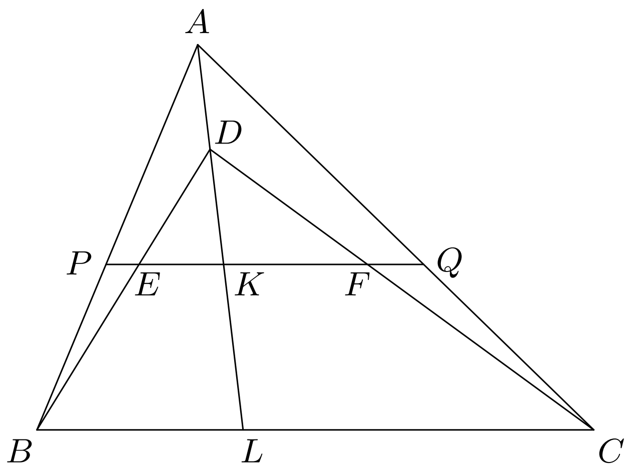
$$\begin{aligned} a_7 &\leq 9 \\ a_6 &= a_7 - b_6 \leq 9 - 3 = 6 \\ a_5 &= a_6 + b_5 \leq 6 - 2 = 4 \\ a_4 &= a_5 + b_4 \leq 4 - 1 = 3 \end{aligned}$$

It follows that no number in S exceeds 96433469. Since it is readily verified that 96433469 is in S, it is the largest number in S. □

7. The common chord of two intersecting circles is always perpendicular to the line joining their centres. All the solutions we present reduce the matter to proving that A lies on the common chord of circles DEQ and DFP. That is, A is on the radical axis of that pair of circles.

Solution 1 (Mel Shu, year 12, Melbourne Grammar School, VIC)

Let the line through A and D intersect PQ at K and BC at L.



The parallel lines imply $\triangle DKE \sim \triangle DLB$ and $\triangle DKF \sim \triangle DLC$. Therefore,

$$\frac{KE}{LB} = \frac{DK}{DL} = \frac{KF}{LC}$$

$$\Rightarrow \qquad \frac{KE}{KF} = \frac{LB}{LC}. \tag{1}$$

We also have $\triangle AKP \sim \triangle ALB$ and $\triangle AKQ \sim \triangle ALC$. Therefore,

$$\frac{KP}{LB} = \frac{AK}{AL} = \frac{KQ}{LC}$$

$$\Rightarrow \qquad \frac{KP}{KQ} = \frac{LB}{LC}. \tag{2}$$

Comparing (1) and (2) we find

$$\frac{KE}{KF} = \frac{KP}{KQ}$$

$$\Rightarrow \quad KE \cdot KF = KQ \cdot KP.$$

Thus K has equal power with respect to circles DEQ and DFP and so the line ADK is the radical axis of the two circles. $\square$

Solution 2 (Alexander Gunning, year 11, Glen Waverley Secondary College, VIC)

Refer to the diagram in solution 1.

The parallel lines imply $\triangle APQ \sim \triangle ABC$. Thus

$$\begin{aligned} & \frac{AP}{AB} = \frac{AQ}{AC} \\ \Rightarrow\quad & 1 - \frac{AP}{AB} = 1 - \frac{AQ}{AC} \\ \Rightarrow\quad & \frac{BP}{AB} = \frac{CQ}{AC}. \end{aligned} \tag{3}$$

Applying Menelaus' theorem to triangle APK with transversal DEB and then again to triangle AQK with transversal DFC we have

$$\frac{KD}{DA} \cdot \frac{AB}{BP} \cdot \frac{PE}{EK} = -1 = \frac{KD}{DA} \cdot \frac{AC}{CQ} \cdot \frac{QF}{FK}.$$

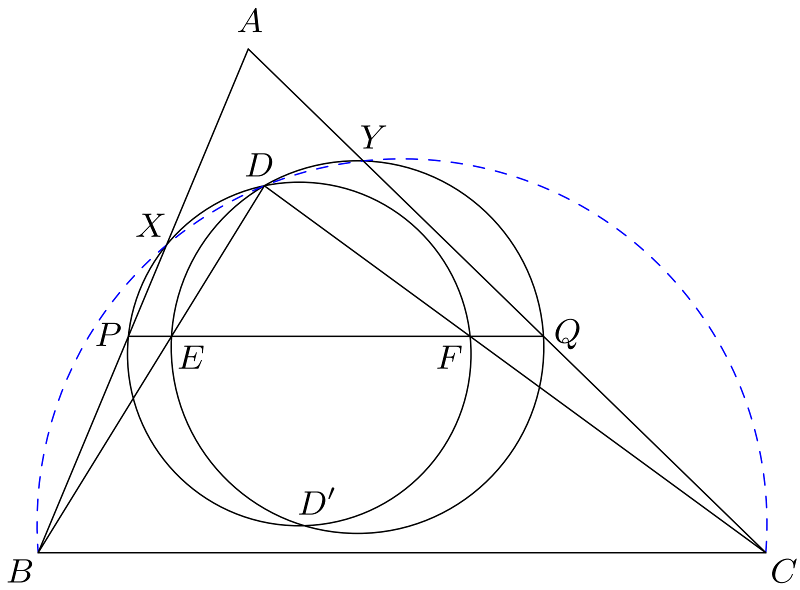
Using (3) we can cancel most of this down to derive

$$\begin{aligned} & \frac{PE}{EK} = \frac{QF}{FK} \\ \Rightarrow\quad & 1 + \frac{PE}{EK} = 1 + \frac{QF}{FK} \\ \Rightarrow\quad & \frac{PK}{EK} = \frac{QK}{FK} \\ \Rightarrow\quad & EK \cdot QK = FK \cdot PK. \end{aligned}$$

Thus K has equal power with respect to circles DEQ and DFP and so the line ADK is the radical axis of the two circles. $\square$

Solution 3 (Seyoon Ragavan, year 10, Knox Grammar School, NSW)

Let circles DFP and DEQ intersect for a second time at point D'. Let circle DFP intersect line AB for the second time at point X and let circle DEQ intersect line AC for the second time at point Y.



Then

$$\begin{aligned} \angle DXA &= DFP && (DXPF \text{ cyclic}) \\ &= \angle DCB && (PQ \parallel BC). \end{aligned}$$

Hence $DXBC$ is cyclic and so X lies on circle DBC. Similarly, Y lies on circle DBC. Thus $DXBCY$ is a cyclic pentagon.

In particular, $XBCY$ is cyclic. From this we have

$$\begin{aligned} \angle AYX &= \angle ABC && (XBCY \text{ cyclic}) \\ &= \angle APQ && (PQ \parallel BC). \end{aligned}$$

Therefore, $XPQY$ is cyclic.

Applying the radical axis theorem to circles $DFPX$, $DEQY$ and $XPQY$ we have that PQ, QY and DD' are concurrent. Since PX and QY intersect at A, we conclude that A lies on the line DD', as required. $\square$

8. **Solution 1** (Jeremy Yip, year 11, Trinity Grammar School, VIC)

Assume at the beginning that k tiles are coloured and $n^2 - k$ tiles are uncoloured. Then the perimeter P of the coloured tiles is at most $4k$. (An edge counts towards the perimeter if it is adjacent to a coloured and an uncoloured tile.)

Every tile Sally colours in reduces the perimeter by 2 or 4 according to whether the newly coloured tile is adjacent to three or four coloured tiles. Therefore, when all the tiles have been coloured, P has been reduced by at least $2(n^2 - k)$. Thus the final perimeter P_{end} satisfies

$$P_{\text{end}} \leq 4k - 2(n^2 - k) = 6k - 2n^2.$$

However, if $k \leq \frac{n^2}{3}$, then $P_{\text{end}} \leq 0$. This is a contradiction because $P_{\text{end}} = 4n$. □

Comment A careful reading of this solution reveals the stronger result $k \geq \frac{n^2+2n}{3}$.

Solution 2 (George Han, year 12,[2] Westlake Boys' High School, NZ)

Let there be initially k coloured tiles and $n^2 - k$ uncoloured tiles. We start giving money to uncoloured tiles as follows.

(i) For each of the k coloured tiles we give \$1 to each of its uncoloured neighbours.

(ii) If an uncoloured tile amasses \$3, we colour it in and give \$1 to to each of its uncoloured neighbours.

If all the tiles are eventually coloured, then all of the $n^2 - k$ tiles, which were originally uncoloured, now each have at least \$3 in them. Thus $D \geq 3(n^2 - k)$ where D is the total amount of dollars at the end.

All dollars in the array come from (i) and (ii). The amount of dollars coming from (i) is at most $4k$. The amount of dollars coming from (ii) is at most $n^2 - k$. Thus $D \leq 4k + n^2 - k$.

Combining the two inequalities for D we deduce

$$\begin{aligned} 4k + n^2 - k &\geq 3(n^2 - k) \\ \Rightarrow \qquad k &\geq \frac{n^2}{3}. \end{aligned}$$

However, since a corner tile has only two neighbours, at least one of the inequalities for D is strict. Thus the final inequality is strict. $\square$

[2]Equivalent to year 11 in Australia.

Solution 3 (Alexander Babidge, year 12, Sydney Grammar School, NSW)

It is convenient for us to use some biology language in this solution. Coloured tiles correspond to organisms, which we shall call *squarelings*. Each unit square of the $n \times n$ array may be occupied by at most one squareling. Furthermore, each squareling has one unit of genes. If k squarelings ($k = 3$ or 4) are adjacent to a vacant square, they produce a *child* in the vacant square. The k squarelings are then said to be *parents* of the child. The child is also a squareling with one unit of genes made up of $\frac{1}{k}$ of a unit of genes from each of its parents.

Each square is adjacent to at most four other squares. Hence at the beginning, before any children are produced, each squareling, which we shall call a *founder*, has the potential to be a parent to at most four children. Since each parent contributes at most one-third of its genes to any child, the total direct gene contribution from any such founder is at most $\frac{4}{3}$.

Consider any squareling that is not a founder. At least three of its neighbouring squares are occupied by its parents. Hence such a squareling has the potential to be the parent of at most one child. Thus the total direct gene contribution from this squareling is at most $\frac{1}{3}$.

Now children can also become parents to other children, but they only pass on genes from their parents. Thus the total gene count from any given founder is at most 1 from itself, $\frac{4}{3}$ from its children, $\frac{1}{3} \cdot \frac{4}{3}$ from its children's children, and so on. If the number of generations is g, then by summing the geometric series, the total gene count from any given founder is at most

$$
\begin{aligned}
1 + \frac{4}{3} + \frac{4}{9} + \cdots + \frac{4}{3^{g-1}} &= 1 + \frac{4}{3}\left(\frac{1 - \frac{1}{3^{g-1}}}{1 - \frac{1}{3}}\right) \\
&< 1 + \frac{4}{3}\left(\frac{1}{1 - \frac{1}{3}}\right) \\
&= 3.
\end{aligned}
$$

If the total number of founders is at most $\frac{n^2}{3}$, then the total gene contribution from these founders is less than n^2, which means that not every square of the array has a squareling in it. This contradiction concludes the proof. $\square$

SCORE DISTRIBUTION/PROBLEM

NUMBER OF STUDENTS/SCORE	PROBLEM NUMBER							
	1	2	3	4	5	6	7	8
0	6	63	58	51	8	30	66	84
1	10	2	5	26	6	22	16	2
2	29	0	14	10	5	7	5	0
3	3	0	4	4	4	12	2	1
4	3	1	2	1	2	7	0	0
5	3	0	1	2	2	4	0	1
6	4	0	2	0	20	10	0	0
7	41	33	13	5	52	7	10	11
AVERAGE MARK	4.2	2.4	1.6	1.1	5.4	2.3	1.0	0.9

AUSTRALIAN MATHEMATICAL OLYMPIAD RESULTS

* indicates a perfect score

** indicates New Zealand school year.

NAME	SCHOOL	YEAR	CERTIFICATE
*Mel Shu	Melbourne Grammar School VIC	12	GOLD
Alex Gunning	Glen Waverley Secondary College VIC	11	GOLD
Jeremy Yip	Trinity Grammar School VIC	11	GOLD
Praveen Wijerathna	James Ruse Agricultural High School NSW	12	GOLD
Seyoon Ragavan	Knox Grammar School NSW	10	GOLD
Yang Song	James Ruse Agricultural High School NSW	11	GOLD
Alex Babidge	Sydney Grammar School NSW	12	GOLD
George Han	Westlake Boys High School NZ	12**	GOLD
Andy Tran	Baulkham Hills High School NSW	12	GOLD
Damon Zhong	Shore School NSW	12	GOLD
Henry Yoo	Perth Modern School WA	11	SILVER
Michael Cherryh	Gungahlin College ACT	11	SILVER
Kevin Xian	James Ruse Agricultural High School NSW	10	SILVER
Vaishnavi Calisa	North Sydney Girls High School NSW	12	SILVER
Richard Gong	Sydney Grammar School NSW	9	SILVER
Matthew Sun	Penleigh and Essendon Grammar School VIC	12	SILVER
Thomas Baker	Scotch College VIC	10	SILVER
Alan Guo	Penleigh and Essendon Grammar School VIC	11	SILVER
Leo Li	Christ Church Grammar School WA	10	SILVER
Michael Chen	Scotch College VIC	11	SILVER
Leo Jiang	Trinity Grammar School NSW	12	SILVER
Jerry Mao	Caulfield Grammar School VIC	8	SILVER
Prince Balanay	Botany Downs Secondary College NZ	12**	SILVER
Allen Gu	Brisbane Grammar School QLD	12	SILVER

Vincent Qi	Auckland International College NZ	13**	SILVER
William Song	Scotch College VIC	10	SILVER
Ivan Zelich	Anglican Church Grammar School QLD	11	SILVER
William Clarke	Sydney Grammar School NSW	12	BRONZE
Allen Lu	Sydney Grammar School NSW	11	BRONZE
Kevin Shen	Saint Kentigern College NZ	11**	BRONZE
Simon Yang	James Ruse Agricultural High School NSW	10	BRONZE
Martin Luk	King's College NZ	12**	BRONZE
Yong See Foo	Nossal High School VIC	10	BRONZE
Linus Cooper	James Ruse Agricultural High School NSW	8	BRONZE
Su Jeong Kim	Rangitoto College NZ	13**	BRONZE
Alexander Barber	Scotch College VIC	10	BRONZE
Alex Ritter	Scotch College VIC	12	BRONZE
Peter Huxford	Newlands College NZ	13**	BRONZE
Austin Zhang	Sydney Grammar School NSW	9	BRONZE
Xuzhi Zhang	Auckland Grammar School NZ	12**	BRONZE
Anand Bharadwaj	Trinity Grammar School VIC	8	BRONZE
Matthew Cheah	Penleigh and Essendon Grammar School VIC	9	BRONZE
Devin He	Christ Church Grammar School WA	10	BRONZE
Sam Bird	Glengunga International High School SA	12	BRONZE
Michelle Chen	Methodist Ladies' College VIC	10	BRONZE
William Chiang	Melbourne Grammar School VIC	12	BRONZE
Andrew Manton-Hall	Sydney Grammar School NSW	12	BRONZE
Eva Wang	Carlingford High School NSW	12	BRONZE