AUSTRALIAN MATHEMATICAL OLYMPIAD

DAY 1
Tuesday, 10 February 2015
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = 4$, $a_2 = 7$, and

$$a_{n+1} = 2a_n - a_{n-1} + 2, \qquad \text{for } n \geq 2.$$

Prove that, for every positive integer m, the number $a_m a_{m+1}$ is a term of the sequence.

2. For each positive integer n, let $s(n)$ be the sum of its digits. We call a number *nifty* if it can be expressed as $n - s(n)$ for some positive integer n.

How many positive integers less than 10,000 are nifty?

3. Let S be the set of all two-digit numbers that do not contain the digit 0. Two numbers in S are called *friends* if their largest digits are equal and the difference between their smallest digits is 1. For example, the numbers 68 and 85 are friends, the numbers 78 and 88 are friends, but the numbers 58 and 75 are not friends.

Determine the size of the largest possible subset of S that contains no two numbers that are friends.

4. Let Γ be a fixed circle with centre O and radius r. Let B and C be distinct fixed points on Γ. Let A be a variable point on Γ, distinct from B and C. Let P be the point such that the midpoint of OP is A. The line through O parallel to AB intersects the line through P parallel to AC at the point D.

(a) Prove that, as A varies over the points of the circle Γ (other than B or C), D lies on a fixed circle whose radius is greater than or equal to r.

(b) Prove that equality occurs in part (a) if and only if BC is a diameter of Γ.

DAY 2
Wednesday, 11 February 2015
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5. Let ABC be a triangle with $\angle ACB = 90°$. The points D and Z lie on the side AB such that CD is perpendicular to AB and $AC = AZ$. The line that bisects $\angle BAC$ meets CB and CZ at X and Y, respectively.

Prove that the quadrilateral $BXYD$ is cyclic.

6. Determine the number of distinct real solutions of the equation

$$(x-1)(x-3)(x-5)\cdots(x-2015) = (x-2)(x-4)(x-6)\cdots(x-2014).$$

7. For each integer $n \geq 2$, let $p(n)$ be the largest prime divisor of n.

Prove that there exist infinitely many positive integers n such that

$$\big(p(n+1)-p(n)\big)\big(p(n)-p(n-1)\big) > 0.$$

8. Let n be a given integer greater than or equal to 3. Maryam draws n lines in the plane such that no two are parallel.

For each equilateral triangle formed by three of the lines, Maryam receives three apples. For each non-equilateral isosceles triangle formed by three of the lines, she receives one apple.

What is the maximum number of apples that Maryam can obtain?