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AUSTRALIAN MATHEMATICAL OLYMPIAD 2016 SOLUTIONS

1. Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = 4$, $a_2 = 7$, and

$$a_{n+1} = 2a_n - a_{n-1} + 2, \qquad \text{for } n \geq 2.$$

Prove that, for every positive integer m, the number $a_m a_{m+1}$ is a term of the sequence.

Solution 1

First, we prove that $a_m = m^2 + 3$ for all positive integers m, by strong induction. Since $a_1 = 1^2 + 3 = 4$ and $a_2 = 2^2 + 3 = 7$, the formula is true for $m = 1$ and $m = 2$. Now suppose that it is true for $m = 1, 2, \ldots, n$ and consider the following calculation.

$$\begin{aligned} a_{n+1} &= 2a_n - a_{n-1} + 2 \\ &= 2(n^2 + 3) - ((n-1)^2 + 3) + 2 \\ &= n^2 + 2n + 4 \\ &= (n+1)^2 + 3 \end{aligned}$$

Note that the second equality here uses the inductive hypothesis. So the formula is true for $m = n + 1$. It follows by strong induction that $a_m = m^2 + 3$ for all positive integers m.

Second, consider the following calculation for a positive integer m.

$$\begin{aligned} a_m a_{m+1} &= (m^2 + 3)((m+1)^2 + 3) \\ &= (m^2 + 3)(m^2 + 2m + 4) \\ &= m^4 + 2m^3 + 7m^2 + 6m + 12 \\ &= (m^2 + m + 3)^2 + 3 \\ &= a_{m^2+m+3} \end{aligned}$$

Since $m^2 + m + 3$ is a positive integer, it follows that $a_m a_{m+1}$ is a term of the sequence.

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Solution 2 (Angelo Di Pasquale, Kevin McAvaney)

Write out the first few terms of the sequence and calculate the differences between successive terms. This results in the conjecture that $a_{m+1} = a_m + 2m + 1$ for all positive integers m, which can be proven by induction. The base case $m = 1$ is straightforward to check. If the statement is true for some $n \geq 1$, then

$$\begin{aligned} a_{n+2} &= 2a_{n+1} - a_n + 2 \\ &= a_{n+1} + (a_n + 2n + 1) - a_n + 2 \qquad \text{(by the inductive assumption)} \\ &= a_{n+1} + 2(n+1) + 1, \end{aligned}$$

which concludes the induction.

We now have

$$\begin{aligned} a_m &= (2m-1) + a_{m-1} \\ &= (2m-1) + (2m-3) + a_{m-2} \\ &\ \vdots \\ &= (2m-1) + (2m-3) + \cdots + 3 + a_1 \\ &= (2m-1) + (2m-3) + \cdots + 3 + 1 + 3 \\ &= m^2 + 3. \end{aligned}$$

The final equality here uses the well-known result that the sum of the first m positive odd integers is equal to m^2. The remainder of the proof is as in the official solution.

Solution 3 (Kevin McAvaney)

The first few differences between successive terms of the sequence are $3, 5, 7, 9, \ldots$. Since these are consecutive odd numbers, this suggests that a_m is a quadratic function of m. So suppose that $a_m = bm^2 + cm + d$. Then the first three terms of the sequence yield the following simultaneous equations.

$$\begin{aligned} 4 &= b + c + d \\ 7 &= 4b + 2c + d \\ 12 &= 9b + 3c + d \end{aligned}$$

By elimination, one obtains $b = 1$, $c = 0$, $d = 3$. Hence, $a_m = m^2 + 3$ and this can be checked by substitution in the given recurrence relation. The remainder of the proof is as in the official solution.

2. For each positive integer n, let $s(n)$ be the sum of its digits. We call a number *nifty* if it can be expressed as $n - s(n)$ for some positive integer n.

How many positive integers less than 10,000 are nifty?

Solution 1

First, let $f(n) = n - s(n)$ and observe that

$$f(10m) = f(10m + 1) = f(10m + 2) = \cdots = f(10m + 9)$$

for all positive integers m. It follows that all nifty positive integers belong to the sequence

$$f(10), f(20), f(30), \ldots.$$

Second, observe that $f(10m) < f(10m + 10)$ for all positive integers m. This statement is equivalent to

$$10m - s(10m) < 10m + 10 - s(10m + 10) \quad \Leftrightarrow \quad s(m + 1) - s(m) < 10,$$

where we have used the fact that $s(10m) = s(m)$. The inequality $s(m + 1) - s(m) < 10$ is trivially true for $m \not\equiv 9 \pmod{10}$, since $m + 1$ and m only differ in their last digit. To see that $s(m + 1) - s(m) < 10$ for $m \equiv 9 \pmod{10}$, suppose that m ends in exactly k 9s. Then the last k digits of $m + 1$ are 0s, while the preceding digit has increased by 1. Therefore, $s(m + 1) - s(m) = 1 - 9k < 10$.

Since $f(10,000) = 9999$ and $f(10,010) = 10,008$, it follows that the nifty positive integers less than 10,000 are precisely the following.

$$f(10) < f(20) < f(30) < \cdots < f(10,000)$$

Therefore, there are 1000 nifty positive integers less than 10,000.

Solution 2 (Kevin McAvaney, Chaitanya Rao, Jamie Simpson)

If n is a non-negative integer, then $n = a_0 + 10a_1 + 100a_2 + 1000a_3 + \cdots$, where $a_0, a_1, a_2, \ldots$ are digits. So all nifty numbers $n - s(n)$ less than 9999 have the form $9a_1 + 99a_2 + 999a_3$.

There are 10 choices for each of a_1, a_2, a_3. We now show that no two such choices yield the same nifty number. Suppose digits u, v, w, x, y, z give $9u + 99v + 999w = 9x + 99y + 999z$. Then

$$(u - x) + 11(v - y) + 111(w - z) = 0.$$

Since $-108 \leq (u - x) + 11(v - y) \leq 108$, we must have $w = z$, which leads to

$$(u - x) + 11(v - y) = 0.$$

Since $-9 \leq (u - x) \leq 9$, we must have $v = y$, which leads to

$$u - x = 0.$$

So there are $10 \times 10 \times 10 = 1000$ nifty numbers in the interval $[0, 9998]$. To obtain nifty numbers in the interval $[1, 9999]$, we simply remove the nifty number 0 and replace it with the nifty number 9999. Therefore, there are 1000 nifty positive integers less than 10,000.

3. Let S be the set of all two-digit numbers that do not contain the digit 0. Two numbers in S are called *friends* if their largest digits are equal and the difference between their smallest digits is 1. For example, the numbers 68 and 85 are friends, the numbers 78 and 88 are friends, but the numbers 58 and 75 are not friends.

Determine the size of the largest possible subset of S that contains no two numbers that are friends.

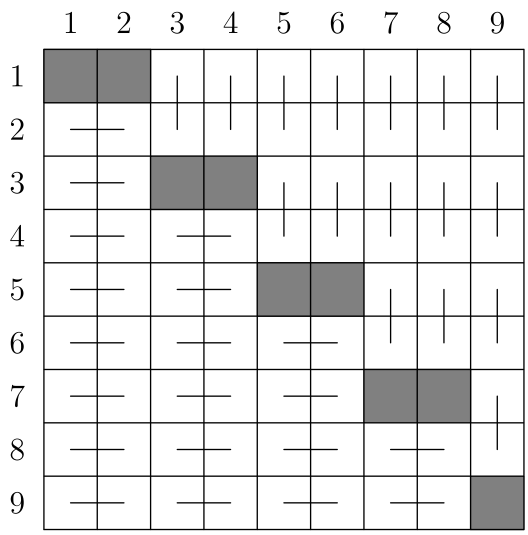
Solution 1 (Andrei Storozhev)

The largest possible subset has 45 elements, which is achieved using the following construction.

- Take the 17 numbers whose smallest digit is 1 — namely, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 31, 41, 51, 61, 71, 81, 91.
- Take the 13 numbers whose smallest digit is 3 — namely, 33, 34, 35, 36, 37, 38, 39, 43, 53, 63, 73, 83, 93.
- Take the 9 numbers whose smallest digit is 5 — namely, 55, 56, 57, 58, 59, 65, 75, 85, 95.
- Take the 5 numbers whose smallest digit is 7 — namely, 77, 78, 79, 87, 97.
- Take the 1 number whose smallest digit is 9 — namely 99.

Now consider the diagram below, where each square represents one of the numbers in S. We consider the square in row i and column j to denote the 2-digit number with tens digit i and units digit j.

Nine of the squares are shaded grey, while the remaining seventy-two squares have been divided into thirty-six pairs.



It is easy to check that each of these pairs denotes two numbers that are friends. Therefore, at most one element of each pair can be in our set. So there certainly cannot be more than $36 + 9 = 45$ elements in our set.

Solution 2 (Angelo Di Pasquale, Ivan Guo, Daniel Mathews, Ian Wanless)

Sort these numbers into rows in the following way, depending on their largest digits.

$$\begin{gathered}11\\12, 22, 21\\13, 23, 33, 32, 31\\14, 24, 34, 44, 43, 42, 41\\15, 25, 35, 45, 55, 54, 53, 52, 51\\\cdots\end{gathered}$$

Two numbers from different rows cannot be friends since their largest digits are different. Two numbers in the same row are friends if and only if they are adjacent. There are $2i-1$ numbers in the ith row. So we can take at most i numbers from the ith row without having an adjacent pair. This is only possible by taking every second number in each row, starting with the first. This yields a maximum of 45 numbers without having a pair of friends. Furthermore, this shows that the maximal construction is unique.

Solution 3 (Joseph Kupka)

Upper bound part of the solution only.

For $i = 2, 4, 6, 8$, the set of 18 numbers with first digit i or $i+1$ can be partitioned into pairs of friends. Therefore, a set without pairs of friends can contain at most 9 elements from such an 18-element set. There are 4 such sets for a total of at most 36 elements. Including all 9 numbers with first digit 1 gives a grand total of at most 45 numbers.

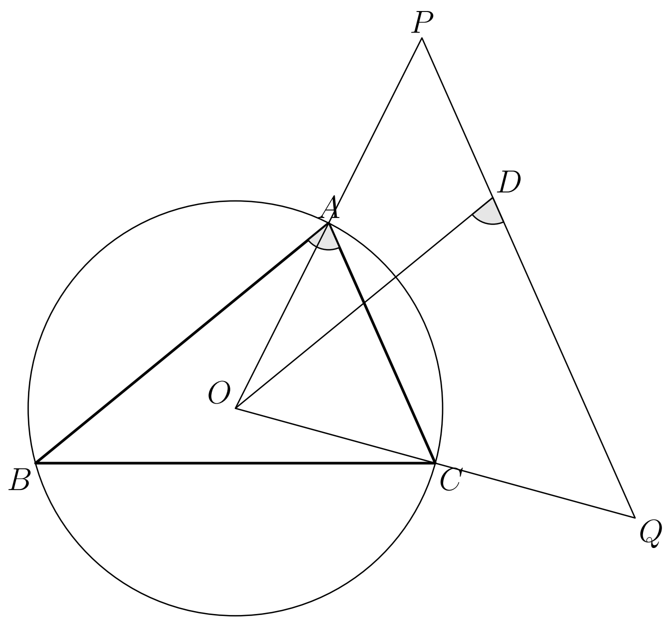
4. Let Γ be a fixed circle with centre O and radius r. Let B and C be distinct fixed points on Γ. Let A be a variable point on Γ, distinct from B and C. Let P be the point such that the midpoint of OP is A. The line through O parallel to AB intersects the line through P parallel to AC at the point D.

   (a) Prove that, as A varies over the points of the circle Γ (other than B or C), D lies on a fixed circle whose radius is greater than or equal to r.

   (b) Prove that equality occurs in part (a) if and only if BC is a diameter of Γ.

Solution 1 (Angelo Di Pasquale)

Let Q be the point such that the midpoint of OQ is C. Observe that neither O nor Q vary with A. Since A and C are the midpoints of OP and OQ, respectively, it follows that AC is parallel to PQ. Hence, D lies on PQ.



Since $OD \parallel BA$ and $DQ \parallel AC$, it follows that $\angle(OD, DQ) = \angle(BA, AC)$, which is fixed, since A lies on Γ. This implies that D lies on a fixed circle through O and Q.

Let this circle through O and Q have centre X and radius s. Since OQ is a chord of this circle of length $2r$, we have $s \geq r$, as required.

Equality occurs if and only if OQ is the diameter of the circumcircle of triangle OQD. This is achieved if and only if $OD \perp DQ$, which is equivalent to $BA \perp AC$. The chords BA and AC are perpendicular if and only if BC is a diameter of Γ.

Solution 2 (Alan Offer)

Let BB' be a diameter of Γ and let E be the point where the line through O parallel to AB meets AC.

Then

$$\begin{aligned}
&\angle(CE, EO) \\
&= \angle(CA, AB) && \text{(since } OE \parallel BA\text{)} \\
&= \angle(CB', B'B) && \text{(since } A, B, B', C \text{ are concyclic)} \\
&= \angle(CB', B'O) && \text{(since } B, B', O \text{ are collinear).}
\end{aligned}$$

It follows that C, B', O and E are concyclic, so E lies on the circumcircle Γ_1 of triangle $CB'O$.

Note that in the limiting case that $C = B'$, we have $\angle(CE, EO) = \angle(CA, AB) = 90°$, so E lies on the circle Γ_1 with diameter CO.

A dilation about O with scale factor 2 fixes the line OE parallel to AB and maps the line AC to the line through P parallel to AC. So the point E is mapped to the point D. Hence, D lies on the circle Γ_2 that is the image of Γ_1 under this dilation.

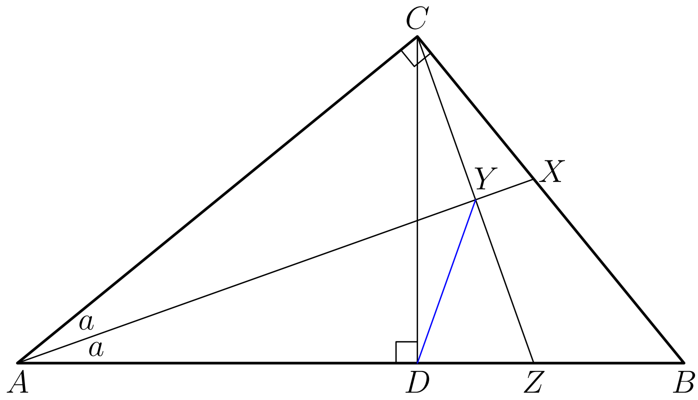
Finally, the radius of Γ_2 is the diameter of Γ_1, which is not greater than r, the length of the chord OC. Equality occurs in the limiting case that $B' = C$, when B and C are diametrically opposite.

5. Let ABC be a triangle with $\angle ACB = 90^\circ$. The points D and Z lie on the side AB such that CD is perpendicular to AB and $AC = AZ$. The line that bisects $\angle BAC$ meets CB and CZ at X and Y, respectively.

   Prove that the quadrilateral $BXYD$ is cyclic.

Solution 1

Let $\angle BAX = \angle CAX = a$. Since triangle ACZ is isosceles with $AC = AZ$ and $\angle ZAC = 2a$, we have $\angle AZC = 90^\circ - a$. Considering the angle sum in the right-angled triangle CDZ allows us to deduce that $\angle DCZ = a$.



We thus have $\angle DCY = \angle DAY = a$, from which it follows that the quadrilateral $DYCA$ is cyclic. Therefore, $\angle CDY = \angle CAY = a$, which implies that $\angle BDY = \angle BDC - \angle CDY = 90^\circ - a$.

Considering the angle sum in the right-angled triangle ABC allows us to deduce that $\angle ABC = 90^\circ - 2a$. Now considering the angle sum in triangle ABX allows us to deduce that $\angle AXB = 90^\circ + a$.

Therefore, $\angle BDY + \angle BXY = (90^\circ - a) + (90^\circ + a) = 180^\circ$ and we conclude that the quadrilateral $BXYD$ is cyclic.

Solution 2 (Ivan Guo)

Since triangle AZC is isosceles, the angle bisector AY is perpendicular to the base CZ. Thus, we have $\angle CYA = \angle CDA = 90^\circ$ and $DYCA$ is cyclic. Also, since $\angle CYX = 90^\circ$, the circumcircle of triangle CXY is tangent to AC at C. By the pivot theorem in triangle ABC, the quadrilateral $BXYD$ is cyclic.

Solution 3 (Chaitanya Rao)

Triangles AZY and ACY are congruent since $AC = AZ$, AY bisects $\angle CAZ$ and side AY is common (SAS). Hence, $\angle ZYA = \angle CYA$ and since C, Y, Z are collinear, $\angle ZYA = \angle CYA = 90^\circ$. Let P be the intersection of CD and AX. Triangles APC and APZ are also congruent as $AC = AZ$, AP bisects $\angle CAZ$ and AP is common (SAS). This implies that

$$\angle AZP = \angle ACP. \tag{1}$$

Note that since $AZ = AC > AD$, the point P lies on the line segment AY. Then since $\angle PDZ + \angle PYZ = 90^\circ + 90^\circ = 180^\circ$, quadrilateral $PYZD$ is cyclic and so

$$\angle DZP = \angle DYP. \tag{2}$$

Hence, we have the following chain of equalities.

$$\begin{aligned}
\angle DYA &= \angle DYP \\
&= \angle DZP && \text{(by equation (2))} \\
&= \angle AZP \\
&= \angle ACP && \text{(by equation (1))} \\
&= \angle ACD \\
&= 90^\circ - \angle CAD && \text{(angle sum in right-angled triangle } ACD\text{)} \\
&= \angle ABC && \text{(angle sum in right-angled triangle } ABC\text{)} \\
&= \angle DBX
\end{aligned}$$

Since an exterior angle $\angle DYA$ is equal to the interior opposite angle $\angle DBX$, we conclude that $BXYD$ is cyclic.

Solution 4 (Ivan Guo)

Observe that AZC is isosceles, so that AY is perpendicular to CZ. Thus, the intersection of CD and AY, say P, is the orthocentre of triangle ACZ. Since PZ and BC are both perpendicular to AC, they must be parallel. Therefore $\angle XBD = \angle PZD = \angle AYD$, where the last equality is due to the fact that $PDZY$ is cyclic. This proves that $BXYD$ is cyclic, as required.

Solution 5 (Kevin McAvaney)

Let $\angle BAX = a$. From the angle sum in triangle ACX, we have $\angle AXB = 90^\circ + a$. Since triangle AZC is isosceles and AY bisects $\angle CAZ$, we know that AY is the perpendicular bisector of CZ. Hence, Y is the circumcentre of triangle CDZ. Therefore, $YD = YZ$ and $\angle YDZ = \angle YZA = 90^\circ - a$. It follows that $\angle AXB + \angle YDZ = 180$, so $BXYD$ is cyclic.

6. Determine the number of distinct real solutions of the equation

$$(x-1)(x-3)(x-5)\cdots(x-2015) = (x-2)(x-4)(x-6)\cdots(x-2014).$$

Solution 1

Let us write $p(x) = a(x) - b(x)$, where

$$a(x) = (x-1)(x-3)(x-5)\cdots(x-2015) \quad \text{and} \quad b(x) = (x-2)(x-4)(x-6)\cdots(x-2014).$$

Note that $a(x)$ is a monic polynomial of degree 1008, while $b(x)$ is a monic polynomial of degree 1007. Therefore, $p(x) = a(x) - b(x)$ is a monic polynomial of degree 1008.

- Suppose that $1 \le m < 2015$ is an integer congruent to 1 modulo 4. Then we have $a(m) = 0$, $b(m) < 0$, $b(m+1) = 0$, and $a(m+1) < 0$. It follows that $p(m) > 0$ and $p(m+1) < 0$, so the intermediate value theorem asserts that there is at least one root of $p(x)$ on the interval $(m, m+1)$.
- Suppose that $1 \le m < 2015$ is an integer congruent to 3 modulo 4. Then we have $a(m) = 0$, $b(m) > 0$, $b(m+1) = 0$, and $a(m+1) > 0$. It follows that $p(m) < 0$ and $p(m+1) > 0$, so the intermediate value theorem asserts that there is at least one root of $p(x)$ on the interval $(m, m+1)$.

These two observations imply that there is at least one root of $p(x)$ on the interval $(m, m+1)$ for $m = 1, 3, 5, \ldots, 2013$. Furthermore, we have $p(2015) = a(2015) - b(2015) = -b(2015) < 0$ and $p(x) \to +\infty$ as $x \to +\infty$. So the intermediate value theorem asserts that there is also at least one root of $p(x)$ on the interval $(2015, \infty)$. So we have found 1008 disjoint intervals on which $p(x)$ has at least one root. Since $p(x)$ has degree equal to 1008, it follows that $p(x)$ must have exactly 1008 real roots.

Solution 2 (Angelo Di Pasquale, Daniel Mathews, Ian Wanless)

As in the official solution, $p(x)$ has degree 1008 and so has at most 1008 real roots. Observe that

$$p(0) = 1 \times 3 \times 5 \times \cdots \times 2015 + 2 \times 4 \times \cdots \times 2014 > 0,$$
$$p(2016) = 1 \times 3 \times 5 \times \cdots \times 2015 - 2 \times 4 \times \cdots \times 2014 > 0.$$

Furthermore, $p(2) < 0$, $p(4) > 0$, $p(6) < 0$, $p(8) > 0$, …, $p(2014) < 0$. So by the intermediate value theorem, each of the 1008 disjoint intervals $(0, 2)$, $(2, 4)$, …, $(2014, 2016)$ contains a root. We conclude that the number of real roots of the polynomial $p(x)$ is 1008.

7. For each integer $n \geq 2$, let $p(n)$ be the largest prime divisor of n.

   Prove that there exist infinitely many positive integers n such that

$$\big(p(n+1) - p(n)\big)\,\big(p(n) - p(n-1)\big) > 0.$$

Solution 1 (Angelo Di Pasquale)

For each integer $n > 2$, we have either $p(n) < p(n-1)$ or $p(n) > p(n-1)$, but $p(n) = p(n-1)$ is impossible. With this in mind, let us define

$$f(n) = \begin{cases} +1, & \text{if } p(n) > p(n-1) \\ -1, & \text{if } p(n) < p(n-1). \end{cases}$$

We are thus required to prove that there exist infinitely many positive integers n such that $f(n) = f(n+1)$.

Assume to the contrary that $f(n+1) = -f(n)$, for all sufficiently large n. For any integer $k > 1$, we have $p(2^k) = 2$ while $p(2^k-1) > 2$, so $f(2^k) = -1$. These imply that $f(2m) = -1$ and $f(2m-1) = +1$, for all sufficiently large m.

Choose $q > 3$ to be a sufficiently large prime. Then by our previous deduction, $f(2q-1) = +1$, $f(2q) = -1$, and $f(2q+1) = +1$. Therefore, $p(2q) < p(2q-1)$ and $p(2q) < p(2q+1)$. However, $p(2q) = q$ and so $2q-1$ and $2q+1$ are both divisible by a prime larger than q. This implies that $2q-1$ and $2q+1$ are both primes.

However, $2q-1$, $2q$, $2q+1$ are three consecutive integers, so one of them is a multiple of 3. This is a contradiction since $2q$ is not divisible by 3, while $2q-1$ and $2q+1$ are primes greater than 3.

Solution 2 (Angelo Di Pasquale)

As in the solution above, consider the function f and deduce that $f(2m) = -1$ and $f(2m-1) = +1$, for all sufficiently large m. Then $f(27^k) = +1$ for all sufficiently large k. The factorisation $27^k - 1 = (27-1)(27^{k-1} + 27^{k-2} + \cdots + 1)$ implies that $p(27^k - 1) \geq 13$, while $p(27^k) = 3$. Therefore, $f(27^k) = -1$, which yields the desired contradiction.

Solution 3 (Mike Clapper, Ivan Guo, Alan Offer)

As in the official solution, we see that if the problem is false, then $p(2m) < p(2m-1)$ and $p(2m) < p(2m+1)$ for all sufficiently large m. So for all sufficiently large k, we have $p(3^k) > p(3^k-1)$ and $p(3^k) > p(3^k+1)$. Since $p(3^k) = 3$, it must be the case that $p(3^k-1) = p(3^k+1) = 2$. This implies that there are two powers of 2 that differ by 2. However, this is not possible for $k > 1$.

Solution 4 (Daniel Mathews)

Define $f(n)$ as in the official solution, and again conclude that $f(2m) = -1$ and $f(2m+1) = 1$ for $m > N$, where N is some positive integer. For sufficiently large m, we have $p(2m+1) > p(2m) = p(m)$ and $p(2m+1) > p(2m+2) = p(m+1)$. Now one of $m, m+1$ is odd, thereby proving the following statement: if n is odd and $n > 2N+1$, then $p(n) > p(r)$ for the odd integer r closest to $\frac{n}{2}$.

Let $S = \{2N+1, 2N+3, \ldots, 4N-1\}$ be the set of odd integers between $2N$ and $4N$. As S contains only odd integers, for each s in S, we have $p(s) \geq 3$. For any odd n with $n > 4N$, we can repeatedly apply the statement above to conclude that $p(n) > p(s)$ for some s in S, so $p(m) > 3$. This implies that for sufficiently large values of M, we have $p(3^M) > 3$, which contradicts the fact that $p(3^M) = 3$.

Solution 5 (Kevin McAvaney) We first note that the greatest common divisor of n and $n+1$ is 1 for all positive integers n. Hence, $p(n)$ and $p(n+1)$ are never equal. We are asked to show that, for infinitely many positive integers n, $p(n-1) < p(n) < p(n+1)$ or $p(n-1) > p(n) > p(n+1)$. If not, then for all sufficiently large n, the sequence of differences $p(n+1) - p(n)$ alternates between positive and negative.

Now for all positive integers k, $p(2^k) = 2$ and $p(2^k - 1) > 2$. Since 2^k is even, $p(2m-1) > p(2m)$ and $p(2m+1) > p(2m)$ for all sufficiently large m. We seek an odd number $2m+1$ that will provide a contradiction. Indeed, for all positive integers k, $p(15^k) = 5$ and $15^k - 1 = (15-1)(1 + 15 + \cdots + 15^{k-1})$. So $p(15^k - 1) \geq 7$, which yields the desired contradiction.

Solution 6 (Ian Wanless)

As per the official solution, we may assume that $f(n+1) = -f(n)$ for all sufficiently large n. Consider an odd prime q. We have $p(q-1) \leq \frac{q-1}{2} < q = p(q)$, so $f(q) = 1$. Since q can be arbitrarily large it follows that there is some N such that $f(n) = 1$ for odd $n > N$ and $f(n) = -1$ for even $n > N$. Now by Dirichlet's theorem, there is a prime $r \equiv 3 \pmod 5$ satisfying $r > N$. For this value of r, we have $p(2r-1) \leq \frac{2r-1}{5} < r = p(2r)$, giving the contradiction that $f(2r) = 1$.

8. Let n be a given integer greater than or equal to 3. Maryam draws n lines in the plane such that no two are parallel.

   For each equilateral triangle formed by three of the lines, Maryam receives three apples. For each non-equilateral isosceles triangle formed by three of the lines, she receives one apple.

   What is the maximum number of apples that Maryam can obtain?

Solution 1 (Andrew Elvey Price)

Define a *based isosceles triangle* to be a triangle formed by three of the lines, with two sides of equal length marked. Then the number of apples obtained by Terence is equal to the number of based isosceles triangles.

For each ordered pair of distinct lines (ℓ_1, ℓ_2), there is at most one isosceles triangle with ℓ_1 unmarked and ℓ_2 marked. So there are at most $\left\lfloor \frac{n-1}{2} \right\rfloor$ based isosceles triangles with ℓ_1 unmarked. Therefore, there are at most $n \times \left\lfloor \frac{n-1}{2} \right\rfloor$ based isosceles triangles altogether.

If n is odd, Terence can achieve this upper bound by drawing the lines to form the sides of a regular polygon with n sides. If n is even, Terence can achieve this upper bound by drawing the lines to form all but one of the sides of a regular polygon with $n+1$ sides.

Solution 2 (Ian Wanless)

Construction part of the solution only.

Take the line along one side from each pair of parallel sides in a regular $2n$-gon. For example, one could use n consecutive sides. Label the lines from 1 up to $2n$ in cyclic order. Then the side labelled i forms the base of an isosceles triangle with the sides labelled $i + c$ and $i - c$ modulo n.