AUSTRALIAN MATHEMATICAL OLYMPIAD

DAY 1
Tuesday, 9 February 2016
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Find all positive integers n such that $2^n + 7^n$ is a perfect square.

2. Let ABC be a triangle. A circle intersects side BC at points U and V, side CA at points W and X, and side AB at points Y and Z. The points U, V, W, X, Y, Z lie on the circle in that order. Suppose that $AY = BZ$ and $BU = CV$.

Prove that $CW = AX$.

3. For a real number x, define $\lfloor x \rfloor$ to be the largest integer less than or equal to x, and define $\{x\} = x - \lfloor x \rfloor$.

(a) Prove that there are infinitely many positive real numbers x that satisfy the inequality
$$\{x^2\} - \{x\} > \frac{2015}{2016}.$$

(b) Prove that there is no positive real number x less than 1000 that satisfies this inequality.

4. A *binary sequence* is a sequence in which each term is equal to 0 or 1. We call a binary sequence *superb* if each term is adjacent to at least one term that is equal to 1. For example, the sequence $0, 1, 1, 0, 0, 1, 1, 1$ is a superb binary sequence with eight terms. Let B_n denote the number of superb binary sequences with n terms.

Determine the smallest integer $n \geq 2$ such that B_n is divisible by 20.

DAY 2
Wednesday, 10 February 2016
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. Find all triples (x, y, z) of real numbers that simultaneously satisfy the equations

$$xy + 1 = 2z$$
$$yz + 1 = 2x$$
$$zx + 1 = 2y.$$

6. Let a, b, c be positive integers such that $a^3 + b^3 = 2^c$.

Prove that $a = b$.

7. Each point in the plane is assigned one of four colours.

Prove that there exist two points at distance 1 or $\sqrt{3}$ from each other that are assigned the same colour.

8. Three given lines in the plane pass through a point P.

(a) Prove that there exists a circle that contains P in its interior and intersects the three lines at six points A, B, C, D, E, F in that order around the circle such that $AB = CD = EF$.

(b) Suppose that a circle contains P in its interior and intersects the three lines at six points A, B, C, D, E, F in that order around the circle such that $AB = CD = EF$. Prove that

$$\frac{1}{2}\,\text{area(hexagon } ABCDEF) \geq \text{area}(\triangle APB) + \text{area}(\triangle CPD) + \text{area}(\triangle EPF).$$

1. Clearly $n = 1$ is obviously a solution. We show that there is no solution for $n > 1$. For reference, in each of the solutions that follow, we suppose that

$$2^n + 7^n = m^2, \tag{1}$$

for some positive integers n and m.

Solution 1 (Sharvil Kesarwani, year 9, Merewether High School, NSW)

Case 1 n is odd and $n > 1$

Considering equation (1) modulo 4 yields

$$\begin{aligned} \text{LHS}(1) &\equiv 0 + (-1)^n \pmod 4 \\ &\equiv 3 \pmod 4. \end{aligned}$$

However, $m^2 \equiv 0$ or $1 \pmod 4$ for any integer m. Thus there are no solutions in this case.

Case 2 n is even

Considering equation (1) modulo 3 yields

$$\begin{aligned} \text{LHS}(1) &\equiv (-1)^n + 1^n \pmod 3 \\ &\equiv 2 \pmod 3. \end{aligned}$$

However, $m^2 \equiv 0$ or $1 \pmod 3$ for any integer m. Thus there are no solutions in this case either.

Having covered all possible cases, the proof is complete. $\square$

Solution 2 (Keiran Lewellen, year 11[1],Te Kura (The Correspondence School), NZ)

Case 1 n is odd and $n > 1$

Let $n = 2k + 1$, where k is a positive integer. Equation (1) can be rewritten as

$$\begin{aligned} 2 \cdot 2^{2k} + 7 \cdot 7^{2k} &= m^2 \\ \Leftrightarrow \qquad 2(2^{2k} - 7^{2k}) &= m^2 - 9 \cdot 7^{2k} \\ &= (m - 3 \cdot 7^k)(m + 3 \cdot 7^k). \end{aligned}$$

The LHS of the above equation is even, so the RHS must be even too. This implies that m is odd. But the both factors on the RHS are even, and so the RHS is a multiple of 4. However, the LHS is not a multiple of 4. So there are no solutions in this case.

Case 2 n is even

Let $n = 2k$, where k is a positive integer. Equation (1) can be rewritten as

$$\begin{aligned} 2^{2k} &= m^2 - 7^{2k} \\ &= (m - 7^k)(m + 7^k). \end{aligned}$$

Hence there exist non-negative integers $r < s$ satisfying the following.

$$\begin{aligned} m - 7^k &= 2^r \\ m + 7^k &= 2^s \end{aligned}$$

Subtracting the first equation from the second yields

$$2^r(2^{s-r} - 1) = 2 \cdot 7^k.$$

Hence $r = 1$, and so $m = 7^k + 2$. Substituting this into equation (1) yields

$$\begin{aligned} 2^{2k} + 7^{2k} &= (7^k + 2)^2 \\ &= 7^{2k} + 4 \cdot 7^k + 4 \\ \Leftrightarrow \qquad 4^k &= 4 \cdot 7^k + 4. \end{aligned}$$

However, this is clearly impossible because the LHS is smaller than the RHS.

Having covered all possible cases, the proof is complete. $\square$

[1]Equivalent to year 10 in Australia.

Solution 3 (Anthony Tew, year 10, Pembroke School, SA)

Case 1 n is odd and $n > 1$

This is handled as in solution 1.

Case 2 n is even

Let $n = 2k$, where k is a positive integer so that equation (1) becomes

$$2^{2k} + 7^{2k} = m^2.$$

Observe that

$$\begin{aligned} 7^{2k} &< 7^{2k} + 2^{2k} \\ &= 7^{2k} + 4^k \\ &< 7^{2k} + 2 \cdot 7^k + 1 \\ &= (7^k + 1)^2. \end{aligned}$$

It follows that $7^k < m < 7^k + 1$. Hence there are no solutions in this case. □

Solution 4 (Xutong Wang, year 10[2], Auckland International College, NZ)

Case 1 n is odd and $n > 1$

This may be handled as in solution 1.

Case 2 n is even

Let $n = 2k$, where k is a positive integer so that equation (1) becomes

$$2^{2k} + 7^{2k} = m^2.$$

Observe that $(2^k, 7^k, m)$ is a primitive a Pythagorean triple.[3] It follows that there exist positive integers $u > v$ such that

$$u^2 - v^2 = 7^k \quad \text{and} \quad 2uv = 2^k.$$

However, the second equation above ensures that $u, v < 2^k$. Hence $u^2 - v^2 < 4^k < 7^k$. Hence there are no solutions in this case. $\square$

[2]Equivalent to year 9 in Australia.

[3]Recall that (x, y, z) is a primitive Pythagorean triple if x, y, and z are pairwise relatively prime positive integers satisfying $x^2 + y^2 = z^2$. All primitive Pythagorean triples take the form

$$(x, y, z) = (u^2 - v^2, 2uv, u^2 + v^2),$$

for some relatively prime positive integers u and v, of opposite parity, and where, without loss of generality, x is odd.

Solution 5 (Zefeng Li, year 9, Camberwell Grammar School, VIC)

Case 1 n is odd and $n > 1$

This is handled as in solution 1.

Case 2 n is even

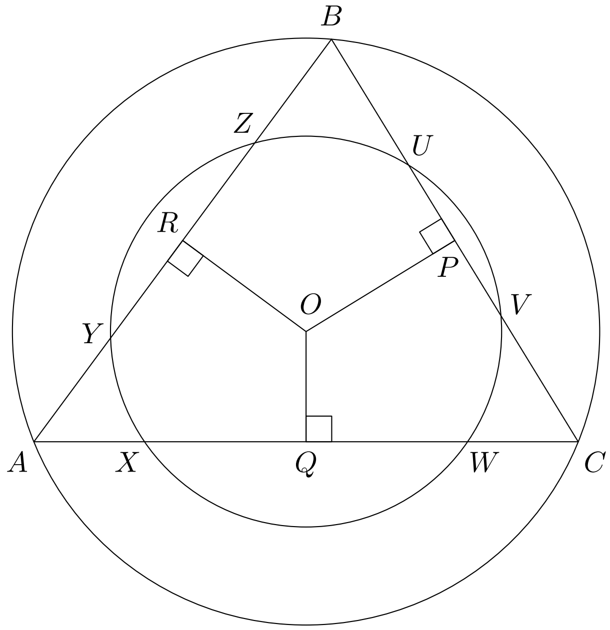
The following table shows the units digit of 2^n, 7^n, and $2^n + 7^n$, for $n = 1, 2, \ldots$. Observe that in each case, the units digit forms a 4-cycle.

n	1	2	3	4	5	...
2^n	2	4	8	6	2	...
7^n	7	9	3	1	7	...
$2^n + 7^n$	9	3	1	7	9	...

Therefore, for n even, the units digit of $2^n + 7^n$ is either 3 or 7. However, the units digit of a perfect square can only be 0, 1, 4, 5, 6, or 9. Hence there are no solutions in this case. $\square$

2. **Solution 1** (Linus Cooper, year 10, James Ruse Agricultural High School, NSW)

Let O be the centre of circle $UVWXYZ$. Let P, Q, and R be the midpoints of UV, WX, and YZ, respectively. Since the perpendicular bisector of any chord of a circle passes through the centre of the circle, we have $OP \perp UV$, $OQ \perp WX$, and $OR \perp YZ$.

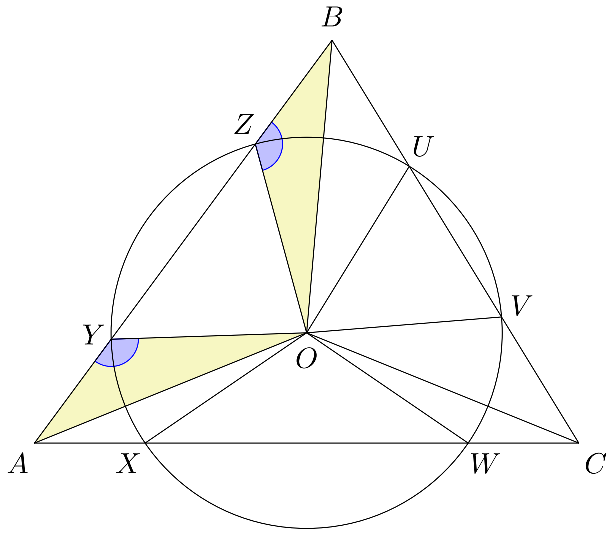


Since R is the midpoint of YZ and $AY = BZ$, we also have that R is the midpoint of AB. Hence OR is the perpendicular bisector of AB. Similarly OP is the perpendicular bisector of BC. Since OR and OP intersect at O, it follows that O is the circumcentre of $\triangle ABC$.

Since O is the circumcentre of $\triangle ABC$, it follows that OQ is the perpendicular bisector of AC. Thus Q is the midpoint of AC. However, since Q is also the midpoint of WX, it follows that $AX = CW$. $\square$

Solution 2 (Jack Liu, year 10, Brighton Grammar School, VIC)

Let O be the centre of circle $UVWXYZ$.



From $OY = OZ$, we have $\angle OYZ = \angle YZO$. Thus

$$\angle AYO = \angle OZB.$$

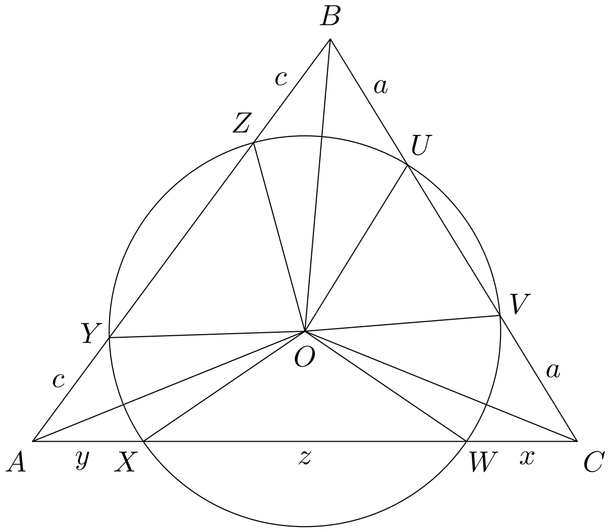
We also have $AY = BZ$. Hence $\triangle AYO \equiv \triangle BZO$ (SAS). Therefore $OA = OB$.

A similar argument shows that $\triangle BUO \equiv \triangle CVO$, and so $OB = OC$.

Hence $OA = OC$. Thus also $\angle XAO = \angle OCW$. From $OX = OW$, we also have $\angle WXO = \angle OWX$, and so $\angle OXA = \angle CWO$. Therefore $\triangle AXO \equiv \triangle CWO$ (AAS). Hence $AX = CW$. $\square$

Solution 3 (James Bang, year 9, Baulkham Hills High School, NSW)

Let $BU = CV = a$, $AY = BZ = c$, $CW = x$, $AX = y$, and $WX = z$.



Considering the power of point A with respect to circle $UVWXYZ$, we have

$$y(y+z) = c(c+YZ).$$

Considering the power of point B with respect to circle $UVWXYZ$, we have

$$c(c+YZ) = a(a+UV).$$

Considering the power of point C with respect to circle $UVWXYZ$, we have

$$a(a+UV) = x(x+z).$$

From the above, it follows that

$$\begin{aligned} & y(y+z) = x(x+z) \\ \Leftrightarrow \quad & x^2 - y^2 = yz - xz \\ \Leftrightarrow \quad & (x-y)(x+y) = -z(x-y) \\ \Leftrightarrow \quad & (x-y)(x+y+z) = 0. \end{aligned}$$

Since $x + y + z = AC > 0$, we have $x = y$. Thus $CW = AX$, as desired. $\square$

3. **Solution 1** (Hadyn Tang, year 7, Trinity Grammar School, VIC)

(a) The following inequality is easily demonstrated by squaring everything.

$$1008 < \sqrt{1008^2+1} < 1008 + \frac{1}{2016}$$

Observe that

$$\lim_{x \to \sqrt{1008^2+1}^-} \{x^2\} = 1$$

and

$$\lim_{x \to \sqrt{1008^2+1}^-} \{x\} < \frac{1}{2016}.$$

Thus there is an open interval $I = \left(r, \sqrt{1008^2+1}\right)$ such that for every $x \in I$ we have $\{x^2\} - \{x\} > \frac{2015}{2016}$. But I contains infinitely many real numbers. $\square$

(b) Suppose that $\{x^2\} - \{x\} > \frac{2015}{2016}$.
Since $\{x^2\} < 1$, we require $\{x\} < \frac{1}{2016}$. If $x < 1000$, then we have

$$n < x < n + \frac{1}{2016}$$

for some $n \in \{0, 1, \ldots, 999\}$. It follows that

$$n^2 < x^2 < \left(n + \frac{1}{2016}\right)^2 = n^2 + \frac{n}{1008} + \frac{1}{2016^2}.$$

Hence

$$\{x^2\} - \{x\} < \{x^2\} < \frac{n}{1008} + \frac{1}{2016^2} \leq \frac{999}{1008} + \frac{1}{2016^2} = \frac{1998 + \frac{1}{2016}}{2016} < \frac{2015}{2016}.$$

Thus no $x < 1000$ can satisfy the given inequality. $\square$

Solution 2 (Charles Li, year 10, Camberwell Grammar School, VIC)

(a) Let $x = n + \frac{1}{m}$ where m and n are positive integers yet to be chosen. We have

$$x^2 = n^2 + \frac{2n}{m} + \frac{1}{m^2}.$$

Thus

$$\{x\} = \frac{1}{m} \quad \text{and} \quad \{x^2\} = \left\{\frac{2n + \frac{1}{m}}{m}\right\}.$$

Consider any fixed integer $m > 20160$. For $n = 1, 2, \ldots$ consider the value of

$$\frac{2n + \frac{1}{m}}{m}.$$

At $n = 1$, this value is less than 1. And every time n increases by 1, this value increases by $\frac{2}{m}$ which is less than $\frac{2}{20160}$. Hence there is a value of n for which

$$1 > \frac{2n + \frac{1}{m}}{m} > \frac{20158}{20160}.$$

For this value of n, we have

$$\{x^2\} - \{x\} = \frac{2n + \frac{1}{m}}{m} - \frac{1}{m} > \frac{20158}{20160} - \frac{1}{20160} > \frac{2015}{2016}.$$

Since such an n can be chosen for each integer $m > 20160$ we have shown the existence of infinitely many x satisfying the inequality. $\square$

(b) Suppose x is a positive real number satisfying

$$\{x^2\} - \{x\} > \frac{2015}{2016}.$$

Let $\lfloor x \rfloor = a$ and $\{x\} = b$. Since $\{x^2\} < 1$, we have $0 < b < \frac{1}{2016}$. We compute

$$\begin{aligned} x^2 &= a^2 + 2ab + b^2 \\ \Rightarrow \quad \{x^2\} &= \{2ab + b^2\}. \end{aligned}$$

Since a and b are non-negative we have

$$2ab + b^2 \geq \{2ab + b^2\} = \{x^2\} > \{x^2\} - \{x\} > \frac{2015}{2016}.$$

Hence

$$b(2a + b) > \frac{2015}{2016}.$$

Since $0 < b < \frac{1}{2016}$ we have

$$2a + b > 2015.$$

And since $0 < b < \frac{1}{2016}$, this implies $a > 1007$. Since a is an integer, we have $a \geq 1008$. Thus $x > 1008$. $\square$

Comment Part (b) of this solution shows that no positive real number x satisfying the inequality can have $\lfloor x \rfloor \leq 1007$. Part (a) of solution 1 shows that the inequality can be satisfied for infinitely many x with $\lfloor x \rfloor = 1008$.

Solution 3 (Jerry Mao, year 10, Caulfield Grammar School, VIC)

(a) Consider $x = 4032n - 1 + \frac{1}{8064}$ for any positive integer n. We compute

$$x^2 = (4032n-1)^2 + n - \frac{2}{8064} + \frac{1}{8064^2}.$$

Hence

$$\begin{aligned} \{x^2\} &= 1 - \frac{2}{8064} + \frac{1}{8064^2} \\ \Rightarrow \quad \{x^2\} - \{x\} &= 1 - \frac{2}{8064} + \frac{1}{8064^2} - \frac{1}{8064} \\ &= \frac{8061}{8064} + \frac{1}{8064^2} \\ &> \frac{8060}{8064} \\ &= \frac{2015}{2016}. \end{aligned}$$

Since this is true for any positive integer n, we have found infinitely many positive x satisfying the required inequality. $\square$

(b) Let $\lfloor x \rfloor = a$ and $\{x\} = b$. Note that $0 \le a < 1000$ and $0 < b < 1$.
As in solution 2, we have $b < \frac{1}{2016}$ and $\{x^2\} = \{2ab + b^2\}$. However since $0 \le a < 1000$ and $0 < b < \frac{1}{2016}$, we have

$$2ab + b^2 < \frac{2000}{2016} + \frac{1}{2016^2} < \frac{2015}{2016}.$$

Hence $\{x^2\} < \frac{2015}{2016}$. It follows that $\{x^2\} - \{x\} < \frac{2015}{2016}$.
Hence no such x can satisfy the required inequality. $\square$

Solution 4 (Zefeng Li, year 9, Camberwell Grammar School, VIC)

(a) Let $x = \underbrace{99\ldots9}_{n}.0001$, where $n \geq 4$. Then $\{x\} = .0001$ and

$$\begin{aligned} x^2 &= \underbrace{99\ldots9}_{n}{}^2 + \underbrace{99\ldots9}_{n} \times 0.0002 + 0.00000001 \\ &= \underbrace{99\ldots9}_{n}{}^2 + 1\underbrace{99\ldots9}_{n-4}.9998 + 0.00000001 \end{aligned}$$

Hence $\{x^2\} > .9998$.
It follows that $\{x^2\} - \{x\} > .9997 > \frac{2015}{2016}$. $\square$

(b) This is done as in solution 3.

Solution 5 (Norman Do, AMOC Senior Problems Committee)

(a) We shall show that $x = n + \frac{1}{n+1}$ satisfies the inequality for all sufficiently large positive integers n.

$$\begin{aligned}
\{x^2\} - \{x\} &= \left\{n^2 + \frac{2n}{n+1} + \frac{1}{(n+1)^2}\right\} - \left\{n + \frac{1}{n+1}\right\} \\
&= \left\{n^2 + 2 - \frac{2}{n+1} + \frac{1}{(n+1)^2}\right\} - \frac{1}{n+1} \\
&= \left(1 - \frac{2}{n+1} + \frac{1}{(n+1)^2}\right) - \frac{1}{n+1} \\
&= 1 - \frac{3}{n+1} + \frac{1}{(n+1)^2} \\
&> 1 - \frac{3}{n+1}
\end{aligned}$$

Therefore, $x = n + \frac{1}{n+1}$ satisfies the inequality as long as n is a positive integer such that

$$1 - \frac{3}{n+1} > \frac{2015}{2016} \quad \Leftrightarrow \quad n > 3 \times 2016 - 1. \qquad \square$$

(b) This is done as in solution 2.

Solution 6 (Wilson Zhao, year 12, Killara High School, NSW)

(a) Let α be any irrational number satisfying $0 < \alpha < \frac{1}{2016}$. For example we could take $\alpha = \frac{1}{2016\sqrt{2}}$. Consider $x = n + \alpha$. We have

$$\begin{aligned} x^2 &= n^2 + 2n\alpha + \alpha^2 \\ \Rightarrow \quad \{x^2\} &= \{2n\alpha + \alpha^2\} \\ \Rightarrow \quad \{x^2\} - \{x\} &= \{2n\alpha + \alpha^2\} - \alpha \end{aligned}$$

If we could arrange for $\{2n\alpha\} < 1 - \alpha^2$, then we would have

$$\{x^2\} - \{x\} = \{2n\alpha\} + \alpha^2 - \alpha.$$

If we could further arrange for $\{2n\alpha\} > \frac{2015}{2016} + \alpha - \alpha^2$, then the corresponding value of x would satisfy the required inequality.

In summary, we would like to ensure the existence of infinitely many positive integers n that satisfy

$$\frac{2015}{2016} + \alpha - \alpha^2 < \{2n\alpha\} < 1 - \alpha^2.$$

However, this is an immediate consequence of the *Equidistribution theorem*[4] applied to the irrational number $\frac{\alpha}{2}$.

(b) This is done as in solution 3.

Comment The full strength of the Equidistribution theorem is not needed in the above proof. The following weaker statement is sufficient.

> For any irrational number r, the sequence $\{r\}, \{2r\}, \{3r\}, \ldots$ takes values arbitrarily close to any real number in the interval $[0, 1)$.

We deem it instructive to provide a proof of the above.

Proof For any positive integer n, consider the intervals

$$I_1 = \left[0, \frac{1}{n}\right], I_2 = \left[\frac{1}{n}, \frac{2}{n}\right], I_3 = \left[\frac{2}{n}, \frac{3}{n}\right], \ldots, I_n = \left[\frac{n-1}{n}, 1\right].$$

By the pigeonhole principle, one of these intervals contains two terms of the sequence $\{r\}, \{2r\}, \{3r\}, \ldots$. Hence there exist positive integers $u \neq v$ such that

$$\begin{aligned} & 0 < \{ur\} - \{vr\} < \frac{1}{n} \\ \Leftrightarrow \quad & 0 < ur - \lfloor ur \rfloor - vr + \lfloor vr \rfloor < \frac{1}{n} \\ \Leftrightarrow \quad & 0 < \{ur - vr\} < \frac{1}{n}. \end{aligned}$$

[4] The theorem states that for any irrational number r, the sequence $\{r\}, \{2r\}, \{3r\}, \ldots$ is uniformly distributed on the interval $[0, 1)$. In particular the sequence takes values arbitrarily close to any real number in the interval $[0, 1)$. See https://en.wikipedia.org/wiki/Equidistribution_theorem

It follows that

$$0 < \{d\} < \frac{1}{n},$$

where $d = (u - v)r$.

If $d > 0$, then $\{d\}, \{2d\}, \{3d\}, \ldots$ is a subsequence of $\{r\}, \{2r\}, \{3r\}, \ldots$. Moreover, it is an arithmetic sequence whose first term is $\{d\}$ and whose common difference is $\{d\}$ up until the point where it is just about to get bigger than 1.

If $d < 0$, then $\{-d\}, \{-2d\}, \{-3d\}, \ldots$ is a subsequence of $\{r\}, \{2r\}, \{3r\}, \ldots$. Moreover, it is an arithmetic sequence whose first term is $1 - \{d\}$ and whose common difference is $-\{d\}$ up until the point where it is just about to get smaller than 0.

In either case we have found a subsequence of $\{r\}, \{2r\}, \{3r\}, \ldots$ which comes within $\{d\}$ of every real number in the interval $[0, 1)$. Since $0 < \{d\} < \frac{1}{n}$, we have shown that the sequence contains a term within $\frac{1}{n}$ of every real number in $[0, 1)$. Since this is true for any positive integer n, the conclusion follows. $\square$

4. **Solution 1** (Matthew Cheah, year 11, Penleigh and Essendon Grammar School, VIC)

Let us say that a binary sequence of length n is *acceptable* if it does not contain two consecutive 0s.[5] Let A_n be the number of acceptable binary sequences of length n.

Lemma The sequence $A_1, A_2, \ldots$ satisfies $A_1 = 2$, $A_2 = 3$, and

$$A_{n+1} = A_n + A_{n-1} \quad \text{for } n = 2, 3, 4 \ldots.$$

Proof It is easy to verify by inspection that $A_1 = 2$ and $A_2 = 3$.

Consider an acceptable sequence S of length $n+1$, where $n \geq 2$.

If S starts with a 1, then this 1 can be followed by any acceptable sequence of length n. Hence there are A_n acceptable sequences in this case.

If S starts with a 0, then the next term must be a 1, and the remaining terms can be any acceptable sequence of length $n-1$. Hence there are A_{n-1} acceptable sequences in this case.

Putting it all together yields, $A_{n+1} = A_n + A_{n-1}$. ☐

Returning to the problem at hand, observe that any superb sequence satisfies the following two properties.

(i) The second and the second last digits are both 1s.

(ii) It contains no subsequence of the form $0, x, 0$.

Moreover, these two properties completely characterise superb sequences. Property (ii), in particular, motivates us to look at every second term a of superb sequence.

Case 1 $n = 2m$ for $m \geq 2$

From (i), the superb sequence is $a_1, 1, a_3, a_4, \ldots, a_{2m-3}, a_{2m-2}, 1, a_{2m}$.

From (ii), the subsequences $a_1 a_3, a_5, \ldots, a_{2m-3}$ and $a_4, a_6, \ldots, a_{2m}$ are both acceptable. The first subsequence has $m-1$ terms as does the second. Hence from the lemma we have $B_{2m} = A_{m-1}^2$.

Case 2 $n = 2m+1$ for $m \geq 3$

From (i), the superb sequence is $a_1, 1, a_3, a_4, \ldots, a_{2m-2}, a_{2m-1}, 1, a_{2m+1}$.

From (ii), the subsequences $a_1, a_3, \ldots, a_{2m+1}$ and $a_4, a_6, \ldots, a_{2m-2}$ are both acceptable. The first subsequence has $m+1$ terms and the second has $m-2$ terms. Hence from the lemma we have $B_{2m+1} = A_{m+1}A_{m-2}$.

Using the lemma it is a simple matter to compute the values of $A_1, A_2, \ldots$ modulo 20 and enter them into the following table. Then the values of B_{2m} and B_{2m+1} are calculated modulo 20 using the two formulas above.

m	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
A_m	2	3	5	8	13	1	14	15	9	4	13	17	$\cdots$
B_{2m}	(1)	4	9	5	4	9	1	16	5	1	16	9	$\cdots$
B_{2m+1}	(3)	(5)	16	19	5	12	15	9	16	15	13	0	$\cdots$

Thus from the table, $n = 25$ is the smallest n such that $20 \mid B_n$. ☐

[5] Here we are forgetting about whether or not a sequence is superb for the time being.

Solution 2 (Linus Cooper, year 10, James Ruse Agricultural High School, NSW)

Call a binary sequence *okay* if it fulfils the superb requirement everywhere except possibly on its last digit. Thus all superb sequences are okay, but there are some okay sequences that are not superb. We classify all okay sequences with two or more terms into the following four types.

Type A: sequences that end in 0,0.

Type B: sequences that end in 0,1.

Type C: sequences that end in 1,0.

Type D: sequences that end in 1,1.

Observe that type A and type B sequences are okay but are not superb, while type C and type D sequences are superb. Let a_n, b_n, c_n, and d_n be the number of okay sequences with n terms of type A, B, C, and D, respectively. Thus $B_n = c_n + d_n$. We seek a recursion for a_n, b_n, c_n, and d_n.

A type A sequence with $n+1$ terms has 0 as its second last term. However, it cannot end in 0,0,0 because this would violate the superbness requirement for its second last digit. Thus $a_{n+1} = c_n$.

A type B sequence with $n+1$ terms has 0 as its second last term. Thus it is built out of a type A or a type C okay sequence with n terms by appending a 1 to any such sequence. Moreover, each such built sequence is okay. Thus $b_{n+1} = a_n + c_n$.

A type C sequence with $n+1$ terms has 1 as its second last term. However, it cannot end in 0,1,0 because this would violate the superbness requirement for its second last digit. Thus $c_{n+1} = d_n$.

A type D sequence with $n+1$ terms has 1 as its second last term. Thus it is built out of a type B or a type D okay sequence with n terms by appending a 1 to any such sequence. Moreover, each such built sequence is okay. Thus $d_{n+1} = b_n + d_n$.

Hence we have a full set of recursive relations for a_n, b_n, c_n, and d_n. We also know that $B_n = c_n + d_n$. By inspection, we have $a_2 = c_2 = 0$ and $b_2 = d_2 = 1$. Thus we can use the following table to calculate a_n, b_n, c_n, d_n, and B_n modulo 20 until we observe that $20 \mid B_n$.

n	2	3	4	5	6	7	8	9	10	11	12	13
a_n	0	0	1	2	2	3	6	10	15	4	0	5
b_n	1	0	1	3	4	5	9	16	5	19	4	5
c_n	0	1	2	2	3	6	10	15	4	0	5	4
d_n	1	2	2	3	6	10	15	4	0	5	4	8
B_n	1	3	4	5	9	16	5	19	4	5	9	12

n	14	15	16	17	18	19	20	21	22	23	24	25
a_n	4	8	13	2	14	15	10	6	15	0	16	17
b_n	9	12	1	15	16	9	5	16	1	15	16	13
c_n	8	13	2	14	15	10	6	15	0	16	17	12
d_n	13	2	14	15	10	6	15	0	16	17	12	8
B_n	1	15	16	9	5	16	1	15	16	13	9	0

From the table, the smallest n with $20 \mid B_n$ is $n = 25$. $\square$

Solution 3 (Based on the solution of Michelle Chen, year 12, Methodist Ladies' College, VIC)

Since no superb sequence can have 0 for its second term or for its second last term, we have following observation.

$$\text{A superb sequence has 1 for its second term and for its second last term.} \tag{1}$$

For each integer $n \geq 2$, let F_n be the set of superb sequences with n terms that end in 0. From observation (1), all such sequences end in 1,0. Let $f_n = |F_n|$.

For each integer $n \geq 2$, let G_n be the set of superb sequences with n terms that end in 1. From observation (1), all such sequences end in 1,1. Let $g_n = |G_n|$.

Consider any superb sequence $s \in F_n$ where $n \geq 3$. Since s ends in 1,0, if we remove the final 0 from s, we end up with a member of G_{n-1}. Conversely, since any member of G_{n-1} ends in 1,1, we may append a 0 to any member of G_{n-1} to create a member of F_n. It follows that $f_n = g_{n-1}$. Since $B_n = f_n + g_n$ we have

$$B_n = g_{n-1} + g_n \quad \text{for } n \geq 3. \tag{2}$$

Consider any superb sequence $s \in G_n$, where $n \geq 5$. The following two sequences are valid possibilities for s.

$$s = \underbrace{1, 1, \ldots 1}_{n} \qquad \text{and} \qquad s = 0, \underbrace{1, 1, \ldots 1}_{n-1}. \tag{3}$$

If s is not one of the two above sequences, we may assume that the last 0 in s appears at the kth position from the left, where $k > 1$, as shown.

$$s = a_1, a_2, \ldots, a_{k-1}, 0, \underbrace{1, 1, \ldots, 1}_{n-k}$$

Note that $k \neq 2$ and $k \neq n-1$, because of observation (1), and $k \neq n$ because of the definition of G_n.

Since $a_{k-2} \neq 0$, we have that $a_1, a_2, \ldots, a_{k-1}$ is a superb sequence with $k-1$ terms. Conversely, from observation (1), any superb sequence with $k-1$ terms can be built into a superb sequence with n terms by appending

$$0, \underbrace{1, 1, \ldots, 1}_{n-k}$$

to it. Since this is valid for any $k \notin \{1, 2, n-1, n\}$, we have

$$g_n = 2 + B_2 + B_3 + \cdots + B_{n-3} \quad \text{for } n \geq 5. \tag{4}$$

Replacing n with $n-1$ in (4) yields

$$g_{n-1} = 2 + B_2 + B_3 + \cdots + B_{n-4} \quad \text{for } n \geq 6. \tag{5}$$

Comparing (4) with (5) we see that

$$g_n = g_{n-1} + B_{n-3} \quad \text{for } n \geq 6.$$

By checking small cases it can be verified that the last equation also holds for $n = 4, 5$.[6] Thus we have

$$\begin{aligned} B_n &= g_{n-1} + g_n \qquad \text{for } n \geq 3, & (2) \\ g_n &= g_{n-1} + B_{n-3} \quad \text{for } n \geq 4. & (6) \end{aligned}$$

It is not hard to eliminate all the g-terms from (2) and (6). For example, replacing n with $n+1$ in (2) and (6) yields

$$\begin{aligned} B_{n+1} &= g_n + g_{n+1} \qquad \text{for } n \geq 2, & (7) \\ g_{n+1} &= g_n + B_{n-2} \quad \text{for } n \geq 3. & (8) \end{aligned}$$

Adding (6) and (8) yields

$$g_n + g_{n+1} = g_{n-1} + g_n + B_{n-2} + B_{n-3} \quad \text{for } n \geq 4.$$

With the help of (2) and (7), the above equation becomes

$$B_{n+1} = B_n + B_{n-2} + B_{n-3} \quad \text{for } n \geq 4. \tag{9}$$

We calculate the values of B_1, B_2, B_3, and B_4 manually by inspection. After this we use the recursion in equation (9) to calculate the values of B_n modulo 20. This is done in the following table.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
B_n	0	1	3	4	5	9	16	5	19	4	5	9	12	1	15	16	9	5	16	1	15	16	13	9	0

Thus the first 0 occurs at $n = 25$, as required. $\square$

[6] Observation (1) speeds this up considerably. In particular, we find $B_1 = 0$, $B_2 = 1$, $g_1 = 0$, $g_2 = 1$, $g_3 = 2$, $g_4 = 2$, and $g_5 = 3$.

Solution 4 (Jerry Mao, year 10, Caulfield Grammar School, VIC)

Let F_n, G_n, f_n, and g_n be as in solution 3. As in solution 3, we have $B_n = g_{n-1} + g_n$ for $n \geq 3$, and that all members of G_n end in 1,1.

Consider a sequence $s \in G_n$ where $n \geq 6$. Either s ends with 1,1,1 or s ends with 0,1,1. But in the second scenario this splits into two further possibilities, namely, that s ends with 1,0,1,1 or s ends with 0,0,1,1.

Case 1 The sequence s ends with 1,1,1.

If we remove the final 1, we end up with a member of G_{n-1}. Conversely, we can append 1 to any member of G_{n-1} to create a member of G_n. So there are g_{n-1} possibilities for this case.

Case 2 The sequence s ends with 1,0,1,1.

We cannot have the sequence 0,1,0 occurring anywhere within a superb sequence. Hence s ends with 1,1,0,1,1. If we remove the final three digits we end up with a member of G_{n-3}. Conversely, we can append 0,1,1 to any member of G_{n-3} to create a member of G_n. So there are g_{n-3} possibilities in this case.

Case 3 The sequence s ends with 0,0,1,1.

We cannot have the sequence 0,0,0 occurring anywhere within a superb sequence. Hence s ends with 1,0,0,1,1. Since 0,1,0 cannot occur anywhere in a superb sequence we see that s ends with $1, 1, 0, 0, 1, 1$. If we remove the final four digits we end up with a member of G_{n-4}. Conversely, we can append 0,0,1,1 to any member of G_{n-4} to create a member of G_n. So there are g_{n-4} possibilities in this case.

It follows from cases 1, 2, and 3, that

$$g_n = g_{n-1} + g_{n-3} + g_{n-4} \quad \text{for all } n \geq 6.$$

By inspection we may calculate that $g_2 = 1$, $g_3 = 2$, $g_4 = 2$, and $g_5 = 3$.

Since $B_n = g_{n-1} + g_n$, we may use the recurrence for g_n to compute successive terms modulo 20, and stop the first time we notice that consecutive terms of the sequence add to a multiple of 20. The sequence $g_2, g_3, g_4, \ldots$ modulo 20 proceeds as follows.

$$1, 2, 2, 3, 6, 10, 15, 4, 0, 5, 4, 8, 13, 2, 14, 15, 10, 6, 15, 0, 16, 17, 12, 8, \ldots$$

We stop because $12 + 8$ is a multiple of 20. This corresponds to $n = 25$. □

The following remarkable solution finds a recursion for B_n directly.

Solution 5 (Ian Wanless, AMOC Senior Problems Committee)

We note that each run of 1s in a superb binary sequence has to have length two or more. For $n \geq 5$ we partition the sequences counted by B_n into three cases.

Case 1 The first run of 1s has length at least three.

In this case, removing one of the 1s in the first run leaves a superb sequence of length $n-1$. Conversely, every superb sequence of length $n-1$ can be extended to a superb sequence of length n by inserting a 1 into the first run of 1s. The process is illustrated via the following example.

$$0, 1, 1, 1, 1, 1, 0, 0, 1, 1, 0 \quad \longleftrightarrow \quad 0, 1, 1, 1, 1, 0, 0, 1, 1, 0$$

So there are B_{n-1} sequences in this case.

Case 2 The first run of 1s has length two and the first term in the sequence is 1.

In this case, the sequence begins with 1,1,0, and what follows is any one of the B_{n-3} superb sequences of length $n-3$.

Case 3 The first run of 1s has length two and the first term in the sequence is 0.

In this case, the sequence begins 0,1,1,0, and what follows is any one of the B_{n-4} superb sequences of length $n-4$.

From cases 1, 2, and 3, we conclude that B_n satisfies the recurrence

$$B_n = B_{n-1} + B_{n-3} + B_{n-4} \quad \text{for } n \geq 5.$$

By checking small cases, we verify that $B_1 = 0$, $B_2 = 1$, $B_3 = 3$, and $B_4 = 4$.

From the above recurrence, we calculate the sequence modulo 20. It begins as

$$0, 1, 3, 4, 5, 9, 16, 5, 19, 4, 5, 9, 12, 1, 15, 16, 9, 5, 16, 1, 15, 16, 13, 9, 0, \ldots$$

from which we deduce that the answer is $n = 25$. ☐

Comment 1 It is possible to determine *all* n for which B_n is a multiple of 20.

We look for repeating cycles for B_n modulo 4 and modulo 5. In light of the recurrence from solution 5, it suffices to find instances of where four consecutive terms in the sequence are equal to another four consecutive terms of the sequence.

Here is a table of values that shows the sequence modulo 4.

n	1	2	3	4	5	6	7	8	9	10
B_n	0	1	3	0	1	1	0	1	3	0

So, modulo 4, the sequence forms a repeating cycle of length 6. It follows that $4 \mid B_n$ whenever $n \equiv 1, 4 \pmod 6$. That is, $n \equiv 1 \pmod 3$.

Here is a table of values that shows the sequence modulo 5.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
B_n	0	1	3	4	0	4	1	0	4	4	0	4	2	1	0	1	4	0	1	1	0	1	3	4

So, modulo 5, the sequence forms a repeating cycle of length 20. It follows that $5 \mid B_n$ whenever $n \equiv 1, 5, 8, 11, 15, 18 \pmod{20}$. That is, $n \equiv 1, 5, 8 \pmod{10}$.

Thus $20 \mid n$ if and only if $n \equiv 1 \pmod 3$ and $n \equiv 1$, 5, or 8 $\pmod{10}$. Solving these congruences shows that $20 \mid n$ if and only if $n \equiv 1$, 25, or 28 $\pmod{30}$.

Hence the smallest integer $n \geq 2$ with $20 \mid B_n$ is $n = 25$. $\square$

Comment 2 We could also find all n for which B_n is a multiple of 20 by further analysing solution 1 as follows.

Recall we had the following pair of equations.

$$B_{2m} = A_{m-1}^2 \tag{1}$$

$$B_{2m+1} = A_{m+1}A_{m-2} \tag{2}$$

Here $A_1, A_2, \ldots$ was defined by $A_1 = 2$, $A_2 = 3$, and $A_{n+1} = A_n + A_{n-1}$ for $n \geq 2$.[7] It is easy to compute that the sequence $A_1, A_2, \ldots$ cycles as

$$\underbracket{0, 1}, 1, \underbracket{0, 1}, 1, \ldots \quad \pmod 2 \tag{3}$$

and

$$\underbracket{2, 3}, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, 1, \underbracket{2, 3}, 0, 3, \ldots \quad \pmod 5. \tag{4}$$

From (3), A_k is even if and only if $k \equiv 1 \pmod 3$. If $n = 2m$ is even, then we need $2 \mid A_{m-1}$. Thus $m \equiv 2 \pmod 3$, and so $n \equiv 2m \equiv 1 \pmod 3$. If $n = 2m + 1$ is odd, then since $A_{m+1} \equiv A_{m-2} \pmod 2$, we need $2 \mid A_{m+1}$. Thus $m \equiv 0 \pmod 3$, and so again $n \equiv 2m + 1 \equiv 1 \pmod 3$. Hence in both cases $n \equiv 1 \pmod 3$.

From (4), $5 \mid A_k$ if and only if $k \equiv 3 \pmod 5$. If $n = 2m$ is even, then we need $5 \mid A_{m-1}$. Thus $m \equiv 4 \pmod 5$, and so $n \equiv 2m \equiv 3 \pmod 5$. So $n \equiv 8 \pmod{10}$. If $n = 2m + 1$ is odd, then we need $5 \mid A_{m+1}$ or $5 \mid A_{m-2}$. So $m \equiv 2$ or $0 \pmod 5$, and hence $n \equiv 2m + 1 \equiv 0$ or $1 \pmod 5$. So $n \equiv 5$ or $1 \pmod{10}$.

Thus $20 \mid B_n$ if and only if $n \equiv 1 \pmod 3$ and $n \equiv 1$, 5, or 8 $\pmod{10}$. This implies that $20 \mid B_n$ if and only if $n \equiv 1$, 25, or 28 $\pmod{30}$.

[7]This is the Fibonacci sequence, except that the indices have been shifted down by 2.

5. **Solution 1** (Jongmin Lim, year 12, Killara High School, NSW)

For reference the equations are

$$xy + 1 = 2z \tag{1}$$
$$yz + 1 = 2x \tag{2}$$
$$zx + 1 = 2y. \tag{3}$$

Subtracting equation (2) from equation (1) yields

$$y(x - z) = -2(x - z) \quad \Leftrightarrow \quad (x - z)(y + 2) = 0.$$

Similarly, subtracting equation (3) from equation (2) yields

$$z(y - x) = -2(y - x) \quad \Leftrightarrow \quad (y - x)(z + 2) = 0.$$

This yields four cases which we now analyse.

Case 1 $x = z$ and $y = x$

Thus $x = y = z$. Putting this into equation (1) yields $x^2 + 1 = 2z$, which is equivalent to $(x - 1)^2 = 0$. Hence $x = y = z = 1$.

Case 2 $x = z$ and $z = -2$

Thus $x = z = -2$. Putting this into (3) immediately yields $y = \frac{5}{2}$.

Case 3 $y = -2$ and $y = x$

Thus $x = y = -2$. Putting this into (1) immediately yields $z = \frac{5}{2}$.

Case 4 $y = -2$ and $z = -2$

Putting this into equation (2) immediately yields $x = \frac{5}{2}$.

So the only possible solutions are $(x, y, z) = (1, 1, 1)$, $\left(\frac{5}{2}, -2, -2\right)$, $\left(-2, \frac{5}{2}, -2\right)$, and $\left(-2, -2, \frac{5}{2}\right)$. It is readily verified that these satisfy the given equations. $\square$

Solution 2 (Jodie Lee, 11, Seymour College, SA)

For reference the equations are

$$xy + 1 = 2z \tag{1}$$
$$yz + 1 = 2x \tag{2}$$
$$zx + 1 = 2y. \tag{3}$$

Subtracting equation (2) from equation (1) yields

$$y(x-z) = -2(x-z) \quad \Leftrightarrow \quad (x-z)(y+2) = 0.$$

This yields the following two cases.

Case 1 $y = -2$

Putting this into equation (3) yields

$$\begin{aligned} zx + 1 &= -4 \\ \Rightarrow \qquad z &= -\frac{5}{x}. \end{aligned}$$

Substituting this into equation (1) yields

$$\begin{aligned} -2x + 1 &= -\frac{10}{x} \\ \Rightarrow \qquad 2x^2 - x - 10 &= 0 \\ \Leftrightarrow \quad (2x-5)(x+2) &= 0. \end{aligned}$$

Thus either $x = -\frac{5}{2}$ which implies $z = -2$, or $x = -2$ which implies $z = -\frac{5}{2}$.

Case 2 $x = z$

Substituting $z = x$ into equation (3) yields

$$\begin{aligned} xy + 1 &= 2x \\ \Rightarrow \qquad y &= \frac{x^2+1}{2}. \end{aligned}$$

Putting this into equation (1) yields

$$\begin{aligned} \frac{x(x^2+1)}{2} + 1 &= 2x \\ \Leftrightarrow \qquad x^3 - 3x + 2 &= 0 \\ \Leftrightarrow \quad (x-1)(x^2+x-2) &= 0 \\ \Leftrightarrow \quad (x-1)(x-1)(x+2) &= 0. \end{aligned}$$

Thus either $x = 1$, which implies $y = z = 1$, or $x = -2$ which implies $z = -2$ and $y = -\frac{5}{2}$.

From cases 1 and 2, the only possible solutions are $(x, y, z) = (1, 1, 1)$, $\left(\frac{5}{2}, -2, -2\right)$, $\left(-2, \frac{5}{2}, -2\right)$, and $\left(-2, -2, \frac{5}{2}\right)$. It is straightforward to check that these satisfy the given equations. $\square$

Solution 3 (Wilson Zhao, year 12, Killara High School, NSW)

For reference the equations are

$$xy + 1 = 2z \tag{1}$$
$$yz + 1 = 2x \tag{2}$$
$$zx + 1 = 2y. \tag{3}$$

Observe that the three equations are symmetric in x, y, and z. So without loss of generality we may assume that $x \geq y \geq z$.

Case 1 $x \geq y \geq z \geq 0$.

It follows that $xy \geq xz \geq yz$. Thus $xy+1 \geq xz+1 \geq yz+1$, and so from equations (1), (2), and (3), it follows that $z \geq y \geq x$. Comparing this with $x \geq y \geq z$ yields $x = y = z$. Then equation (1) becomes $x^2 + 1 = 2x$, that is $(x-1)^2 = 0$. Therefore $x = 1$, and so $(x, y, z) = (1, 1, 1)$ in this case.

Case 2 $x \geq y \geq 0 > z$

It follows that $0 > 2z = xy + 1 > 0$, a contradiction. So this case does not occur.

Case 3 $x \geq 0 > y \geq z$

It follows that $xy \geq xz$. Thus $xy + 1 \geq xz + 1$, and so from equations (1) and (3), we have $z \geq y$. Comparing this with $y \geq z$ yields $y = z$. Substituting this into (2) and rearranging yields

$$x = \frac{z^2+1}{2}.$$

Substituting this into (1) and also remembering that $y = z$ yields

$$\begin{aligned}
& \frac{(z^2+1)z}{2} + 1 = 2z \\
\Leftrightarrow \quad & z^3 - 3z + 2 = 0 \\
\Leftrightarrow \quad & (z-1)(z^2+z-2) = 0 \\
\Leftrightarrow \quad & (z-1)(z-1)(z+2) = 0.
\end{aligned}$$

This has the two solutions $z = 1$ and $z = -2$. Since $z < 0$ we have $z = -2$. It follows that $y = -2$ and $x = \frac{5}{2}$. Hence $(x, y, z) = \left(\frac{5}{2}, -2, -2\right)$ in this case.

Case 4 $0 > x \geq y \geq z$

It follows that $0 > 2x = yz + 1 > yz > 0$, a contradiction. So this case does not occur.

From the four cases thus analysed we have the possibilities $(x, y, z) = (1, 1, 1)$ or any of the three three permutations of $\left(\frac{5}{2}, -2, -2\right)$. It is readily verified that these do indeed satisfy the original equations. $\square$

Solution 4 (Yong See Foo, year 12, Nossal High School, VIC)

For reference the equations are

$$\begin{aligned} xy + 1 &= 2z && (1)\\ yz + 1 &= 2x && (2)\\ zx + 1 &= 2y. && (3) \end{aligned}$$

Let $p = xyz$. Multiplying equation (1) by z, yields

$$\begin{aligned} xyz + z &= 2z^2 \\ \Rightarrow \quad 2z^2 + z + p &= 0 \end{aligned}$$

Similarly, we find

$$2x^2 + x + p = 0 \quad \text{and} \quad 2y^2 + y + p = 0.$$

Thus x, y, and z are all roots of the same quadratic equation. From the quadratic formula we have that each of x, y, and z is equal to

$$\frac{1 - \sqrt{1+8p}}{4} \quad \text{or} \quad \frac{1 + \sqrt{1+8p}}{4}. \tag{4}$$

Case 1 Not all of x, y, and z are equal.

Since the given equations are symmetric in x, y, and z, we can assume without loss of generality that $x \neq y$, and so $p \neq 0$. It follows that the product xy is equal to the product of the two expressions found in (4). Hence

$$xy = \left(\frac{1 - \sqrt{1+8p}}{4}\right)\left(\frac{1 + \sqrt{1+8p}}{4}\right) = -\frac{p}{2}.$$

However, since $p = xyz \neq 0$, it follows that $z = -2$. But z is also equal to one of the expressions found in (4). The second expression in (4) is definitely positive, while $z = -2 < 0$. Thus we have

$$-2 = \frac{1 - \sqrt{1+8p}}{4}.$$

Solving for p yields $p = 10$. Substituting this into (4) yields $\{x, y\} = \left\{\frac{5}{2}, -2\right\}$. Thus (x, y, z) can be any permutation of $\left(\frac{5}{2}, -2, -2\right)$.

Case 2 $x = y = z$

Equation (1) becomes $x^2 + 1 = 2x$. This is the same as $(x-1)^2 = 0$. Therefore $x = 1$, and so $(x, y, z) = (1, 1, 1)$ in this case.

It is readily verified that $(1, 1, 1)$ and the three permutations of $\left(\frac{5}{2}, -2, -2\right)$ satisfy the original equations. $\square$

6. **Solution 1** (Based on the solution of Shivasankaran Jayabalan, year 10, Rossmoyne Senior High School, WA)

For reference we are given

$$a^3 + b^3 = 2^c. \tag{1}$$

Assume for the sake of contradiction that there are positive integers $a \neq b$ such that $a^3 + b^3$ is a power of two. Of all such solutions choose one with $|a - b|$ minimal.

Case 1 Both a and b are even.

Let $x = \frac{a}{2}$ and $y = \frac{b}{2}$. Then $x^3 + y^3$ is also a power of two. But $0 < |x - y| < |a - b|$, which contradicts the minimality of $|a - b|$.

Case 2 One of a and b is even while the other is odd.

It follows that $a^3 + b^3$ is odd. But the only odd power of two is 1. Hence $a^3 + b^3 = 1$. However this is impossible for positive integers a and b.

Case 3 Both of a and b are odd.

Equation (1) may be rewritten as

$$(a + b)(a^2 - ab + b^2) = 2^c.$$

Since both a and b are odd, then so is $a^2 - ab + b^2$. However the only odd factor of a power of two is 1. Hence

$$\begin{aligned} & a^2 - ab + b^2 = 1 \\ \Leftrightarrow \quad & (a - b)^2 + ab = 1. \end{aligned}$$

However, this is also impossible because $(a - b)^2 \geq 1$ and $ab \geq 1$. $\square$

Solution 2 (Yasiru Jayasooriya, year 8, James Ruse Agricultural High School, NSW)

For reference we are given

$$a^3 + b^3 = 2^c. \tag{1}$$

Let 2^n be the highest power of two dividing both a and b. Thus we may write

$$a = 2^n A \quad \text{and} \quad b = 2^n B \tag{2}$$

for some positive integers A and B such that at least once of A and B is odd.

From this we see that $2^{3n} \mid 2^c$, so that $c \geq 3n$. Let $d = c - 3n$. Substituting (2) into (1) and tidying up yields

$$A^3 + B^3 = 2^d. \tag{3}$$

Since A and B are positive integers, we have $A^3 + B^3 \geq 2$, and so $d \geq 1$. Thus A^3 and B^3 have the same parity. Hence A and B have the same parity. Since at most one of A and B is even, it follows that both A and B are odd.

Factoring the LHS of (3) yields

$$(A + B)(A^2 - AB + B^2) = 2^d.$$

Since A and B are both odd it follows that $A^2 - AB + B^2$ is odd. But the only odd factor of 2^d is 1. Thus we have

$$A^2 - AB + B^2 = 1 \quad \text{and} \quad A + B = 2^d.$$

From (3), this implies

$$A^3 + B^3 = A + B. \tag{4}$$

However $x^3 > x$ for any integer $x > 1$. So if (4) is true, we must have $A = B = 1$. It follows that $a = b = 2^n$. $\square$

Solution 3 (Barnard Patel, year 12[8], Wellington College, NZ)

For reference we are given

$$a^3 + b^3 = 2^c. \tag{1}$$

This may be rewritten as

$$(a+b)(a^2 - ab + b^2) = 2^c.$$

It follows that both $a+b$ and $a^2 - ab + b^2$ are powers of two. Consequently, we have the following equations for some non-negative integers d and e.

$$a + b = 2^d \tag{2}$$

$$a^2 - ab + b^2 = 2^e \tag{3}$$

If we square equation (2) and then subtract equation (3), we find

$$3ab = 2^{2d} - 2^e. \tag{4}$$

The LHS of (4) is positive, hence so also is the RHS. Thus $2d > e$, and so

$$3ab = 2^e(2^{2d-e} - 1).$$

Since $\gcd(3, 2^e) = 1$, we have $2^e \mid ab$. From (3), this is $a^2 - ab + b^2 \mid ab$. Thus

$$\begin{aligned} & a^2 - ab + b^2 \le ab \\ \Leftrightarrow \quad & (a-b)^2 \le 0. \end{aligned}$$

Since squares are non-negative, it follows that $a = b$, as desired. □

[8]Equivalent to year 11 in Australia.

Solution 4 (Keiran Lewellen, year 11[9], Te Kura (The Correspondence School), NZ)

For reference we are given

$$a^3 + b^3 = 2^c. \tag{1}$$

Without loss of generality $a \geq b$.

We are given that a and b are positive integers. Hence $a^3 + b^3 = 2^c$ is a power of two that is greater than 1. Thus $c \geq 1$ and a^3 and b^3 have the same parity. It follows that a and b have the same parity. Hence we may let $a + b = 2x$ and $a - b = 2y$ for integers $0 \leq y < x$. It follows that

$$a = x + y \quad \text{and} \quad b = x - y.$$

Substituting this in to (1) yields

$$\begin{aligned} & 2x^3 + 6xy^2 = 2^c \\ \Rightarrow \quad & x(x^2 + 3y^2) = 2^{c-1}. \end{aligned}$$

Consequently, we have the following for some non-negative integers r and s.

$$x = 2^r \tag{2}$$
$$x^2 + 3y^2 = 2^s \tag{3}$$

Note that $2r \leq s$ because $x^2 \leq 2^s$. Substituting (2) into (3) and rearranging yields

$$3y^2 = 2^{2r}(2^{s-2r} - 1).$$

Since $\gcd(3, 2^{2r}) = 1$, we have $2^{2r} \mid y^2$, and so $2^r \mid y$. Thus $x \mid y$. But $0 \leq y < x$. Hence $y = 0$, and so $a = b$. $\square$

[9] Equivalent to year 10 in Australia.

Solution 5 (Anthony Tew, year 10, Pembroke School, SA)

For reference we are given

$$a^3 + b^3 = 2^c. \tag{1}$$

This may be rewritten as

$$(a+b)(a^2 - ab + b^2) = 2^c.$$

Suppose for the sake of contradiction that $a \neq b$. Without loss of generality $a < b$. Let $b = a + k$ for some positive integer k. Substituting this into the above equation and tidying up yields

$$(2a+k)(a^2 + ak + k^2) = 2^c.$$

Consequently, we have the following for some non-negative integers d and e.

$$2a + k = 2^d \tag{2}$$

$$a^2 + ak + k^2 = 2^e \tag{3}$$

Squaring (2) yields

$$4a^2 + 4ak + k^2 = 2^{2d}. \tag{4}$$

Now observe that

$$4a^2 + 4ak + k^2 < 4(a^2 + ak + k^2) < 4(4a^2 + 4ak + k^2).$$

This is the same as

$$2^{2d} < 2^{e+2} < 2^{2d+2}.$$

Hence $e + 2 = 2d + 1$. That is, $e = 2d - 1$ Then comparing (3) and (4) yields

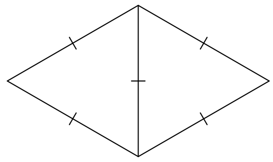
$$\begin{aligned} 4a^2 + 4ak + k^2 &= 2(a^2 + ak + k^2) \\ \Leftrightarrow \quad k^2 - 2ak - 2k^2 &= 0 \\ \Leftrightarrow \quad (k-a)^2 &= 3k^2. \end{aligned}$$

But $3k^2$ can never be a perfect square for any positive integer k. This contradiction concludes the proof. $\square$

7. **Solution 1** (Matthew Cheah, year 11, Penleigh and Essendon Grammar School VIC)

Suppose, for the sake of contradiction, that we can assign every point in the plane one of four colours in such a way that no two of points at distance 1 or $\sqrt{3}$ from each other are assigned the same colour.

Consider any rhombus formed by joining two unit equilateral triangles together as shown. Call such a figure a 60°-*rhombus*.

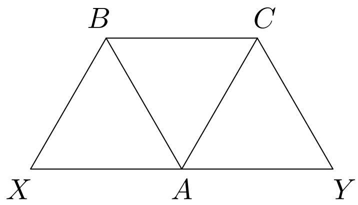


Lemma 1 The four vertices of a 60°-rhombus are assigned different colours.

Proof It is easily computed that the long diagonal of the rhombus has length $\sqrt{3}$. All other distances between pairs of vertices of the rhombus are of length 1. Thus no two vertices can be assigned the same colour. $\square$

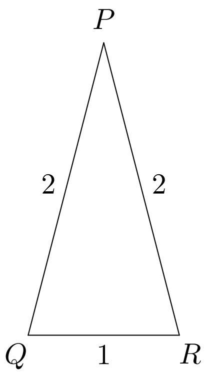
Lemma 2 Any two points at distance 2 from each other are assigned the same colour.

Proof Consider any two points X and Y such that $XY = 2$. We can build a net of unit equilateral triangles as follows.



From lemma 1, points A, C, B, and X are assigned four different colours. Again from lemma 1, points A, B, C, and Y are assigned four different colours. But since only four colours are available, X and Y must be assigned the same colour. $\square$

To finish the proof of the given problem, consider the following triangle.



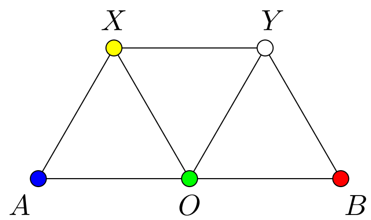
From lemma 2, points P and Q are assigned the same colour. Again from lemma 2, points P and R are assigned the same colour. Hence Q and R are assigned the same colour. This is a contradiction because $QR = 1$. $\square$

Solution 2 (Angelo Di Pasquale, Director of Training, AMOC)

Suppose that the four colours are blue, red, yellow, and green. We argue by contradiction and suppose that there do not exist two points at distance 1 or $\sqrt{3}$ from each other that have the same colour.

Consider an isosceles triangle ABC with $BC = 1$ and $AB = AC = 2$. Since B and C must be different colours, one of them is coloured differently to A. Without loss of generality, we may suppose that A is blue and B is red.

Let us orient the plane so that AB is a horizontal segment.



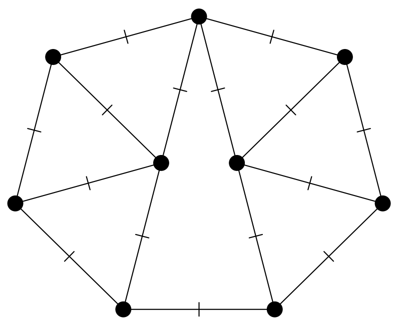
Let O be the midpoint of AB. Then as $AO = BO = 1$, it follows that O is not blue or red. Without loss of generality we may suppose that O is green.

Let X be the point above the line AB such that $\triangle AOX$ is equilateral. It is easy to compute that $XB = \sqrt{3}$ and $XA = XO = 1$. Hence, X is not red, blue or green. Thus X must be yellow.

Finally, let Y be the point above the line AB such that $\triangle BOY$ is equilateral. Then it is easy to compute that $YX = YO = YB = 1$ and $YA = \sqrt{3}$. Hence Y cannot be any of the four colours, giving the desired contradiction. ☐

Comment (Kevin Xian, year 12, James Ruse Agricultural High School, NSW)

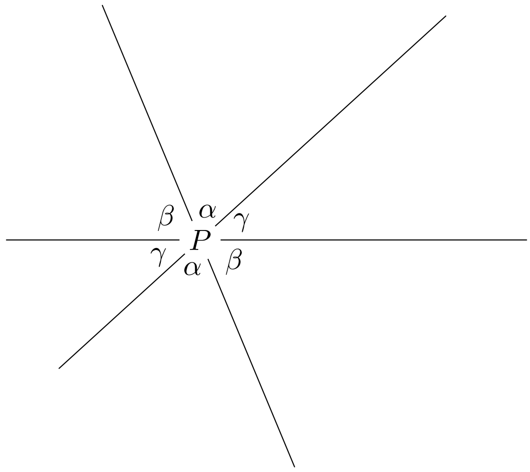
Forget about trying to assign colours to *every* point in the plane! An analysis of either of the presented solutions shows that if one tries to colour the nine points shown in the diagram below using four colours, then there will always be two points of the same colour at distance 1 or $\sqrt{3}$ from each other.



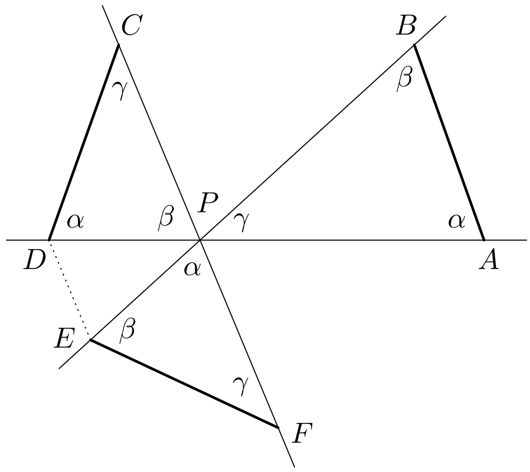
8. This was the most difficult problem of the 2016 AMO. Just six contestants managed to solve it completely.

8 (a). **Solution 1** (Ilia Kucherov, year 12, Westall Secondary College, VIC)

Let α, β, and γ be the angles between the lines as shown in the diagram on the left below.



Note that $\alpha+\beta+\gamma=180°$. This permits us to locate segments AB, CD, and EF, all of unit length, as shown in the diagram on the right below. We shall prove that $ABCDEF$ is a cyclic hexagon.



Since $\angle BAD = \angle ADC$ and $AB = CD$, it follows that $ABCD$ is an isosceles trapezium with $AD \parallel BC$. Thus $ABCD$ is a cyclic quadrilateral.[10]

Similarly, $CDEF$ is cyclic with $CF \parallel DE$. Thus

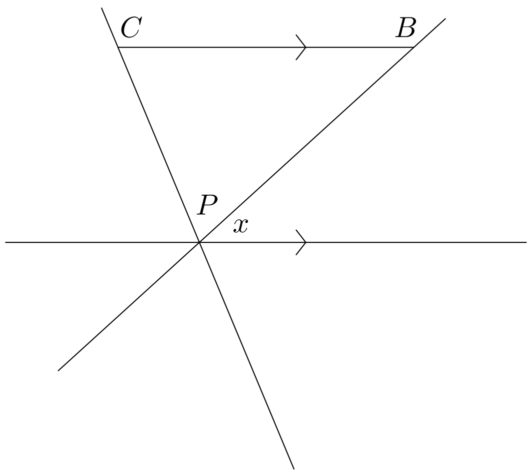
$$\angle BED = \angle BPC = \alpha = \angle BAD.$$

Hence $ABDE$ is cyclic. Since quadrilaterals $ABCD$ and $ABDE$ are both cyclic, it follows that $ABCDE$ is a cyclic pentagon. But then since $ABCDE$ and $CDEF$ are both cyclic, it follows that $ABCDEF$ is a cyclic, as desired. □

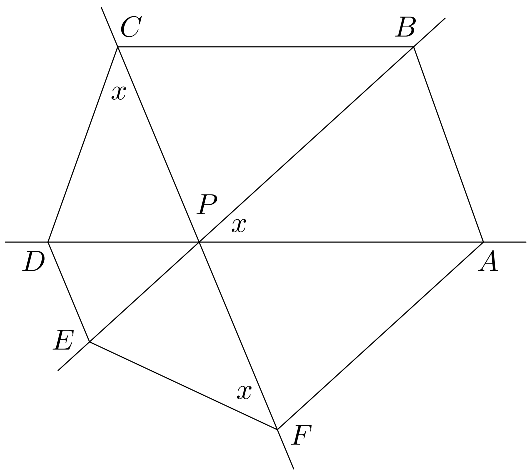
[10] Every isosceles trapezium is cyclic.

Solution 2 (Matthew Cheah, year 11, Penleigh and Essendon Grammar School, VIC)

Orient the diagram so that one of the lines is horizontal. Choose any points B and C, one on each of the other two lines, such that BC is parallel to the horizontal line. Let x be the angle as shown in the diagram.



Let D be the point on the horizontal line such that $\angle DCP = x$.[11] Let E be the point on the line BP such that $DE \parallel CP$. Let F be the point on the line CP such that $\angle PFE = x$.[12] Let A be the point on the line DP such that $AF \parallel BE$. We claim that $ABCDEF$ is cyclic.



From $DE \parallel CP$, we have $\angle EDC = 180° - x$. And from $AD \parallel BC$, we have $\angle CBE = x$. So $BCDE$ is cyclic because $\angle CBE + \angle EDC = 180°$.

Quadrilateral $CDEF$ is cyclic because $\angle EDC + \angle CFE = 180°$.

From $DE \parallel CF$, we have $\angle FED = 180° - x$. And from $AF \parallel BE$, we have $\angle DAF = x$. So $DEFA$ is cyclic because $\angle FED + \angle DAF = 180°$.

Since $BCDE$, $CDEF$, and $DEFA$ are all cyclic, it follows that $ABCDEF$ is cyclic, as claimed. $\square$

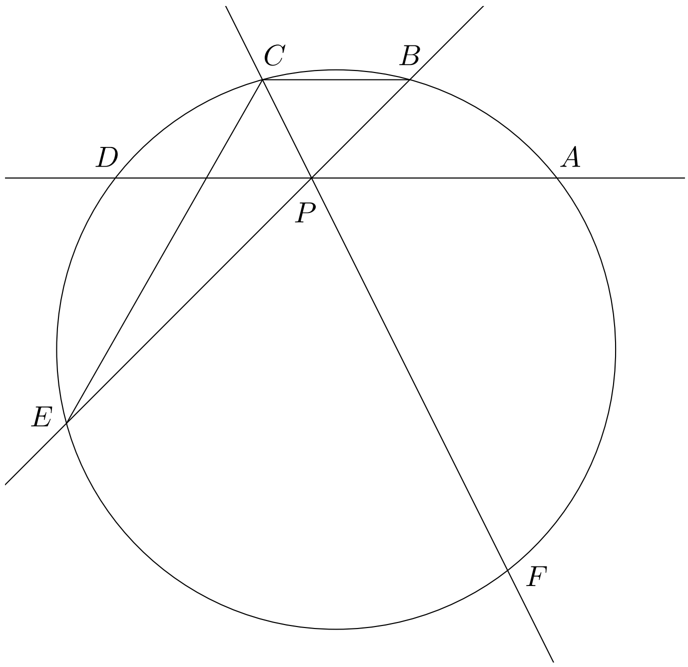
[11] Note that $\angle APC > x$. So the point D really does exist and lies on the same side of the line BP as C as shown in the diagram.

[12] Note that $\angle CPE = \angle CPD + x > x$. So the point F really does exist and has the property that P lies between C and F as shown in the diagram.

Solution 3 (Angelo Di Pasquale, Director of Training, AMOC)

As in solution 2, orient the diagram so that one of the lines is horizontal, and choose any points B and C, one on each of the other two lines, such that BC is parallel to the horizontal line.

Let E be a variable point on line BP so that P lies between B and E. Consider the family of circles passing through points B, C, and E. Let A, D, and F be the intersection points of circle BCE with the three given lines so that A, B, C, D, E, and F are in that order around the circle. Then $BC \parallel AD$. Thus $ABCD$ is an isosceles trapezium with $AB = CD$.



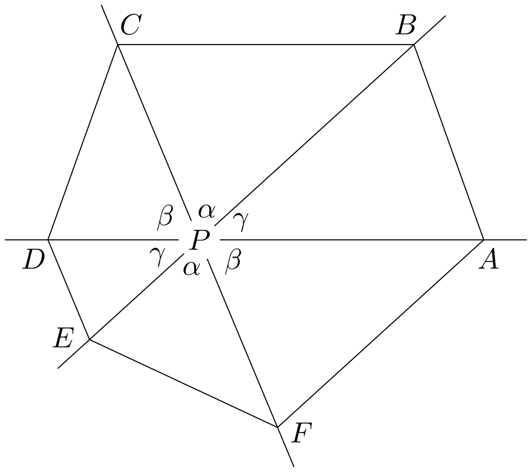
Consider the ratio $r = \frac{EF}{AB}$ as E varies on the line BP so that P lies between B and E. As E approaches P, the length AB approaches $\min\{BP, CP\}$ while the length EF approaches 0. Hence, r approaches 0.

As E diverges away from P, we have $\angle ADB$ approaches $0°$, while $\angle ECF$ approaches $\angle BPC$. Thus, eventually $\angle ECF > \angle ADB$, and so $r > 1$.

Since r varies continuously with E, we may apply the intermediate value theorem to deduce that there is a position for E such that $r = 1$. The circle BCE now has the required property because $EF = AB = CD$. $\square$

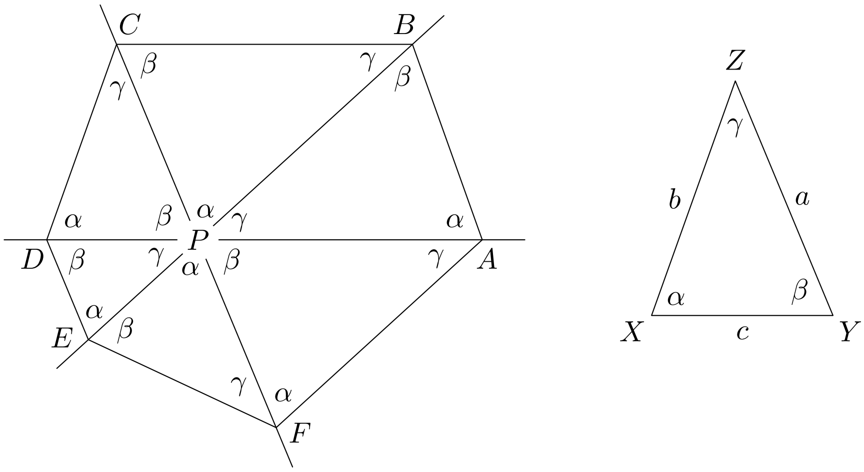
8 (b). **Solution 1** (Seyoon Ragavan, year 12, Knox Grammar School, NSW)

Let $AB = CD = EF = r$ and label the angles between the lines as shown.



Note that $ABCD$ is an isosceles trapezium because it is cyclic and has $AB = CD$. Thus $AD \parallel BC$. Hence $\angle CBP = \gamma$ and $\angle PCB = \beta$. Similarly we have $BE \parallel AF$ and $CF \parallel DE$, which imply $\angle AFP = \alpha$, $\angle PAF = \gamma$, $\angle EDP = \beta$, and $\angle PED = \alpha$.

Since $ABCDEF$ is cyclic we have $\angle ADC = \angle AFC = \alpha$, $\angle DCF = \angle DAF = \gamma$, $\angle EBA = \angle EDA = \beta$, $\angle BAD = \angle BED = \alpha$, $\angle CFE = \angle CBE = \gamma$, and $\angle FEB = \angle FCB = \beta$.



Let $\triangle XYZ$ be a reference triangle that satisfies $\angle YXZ = \alpha$, $\angle ZYX = \beta$, and $\angle XZY = \gamma$. Let $YZ = a$, $ZX = b$, and $XY = c$. We have

$$\triangle ABP \sim \triangle PCB \sim \triangle DPC \sim \triangle EDP \sim \triangle PEF \sim \triangle FPA \sim \triangle XYZ.$$

Let $S = |XYZ|$.[13] From $\triangle ABP \sim \triangle XYZ$, we deduce

$$\frac{|ABP|}{|XYZ|} = \left(\frac{AB}{XY}\right)^2 \quad \Rightarrow \quad |APB| = \frac{r^2 S}{c^2}$$

and

$$\frac{AP}{XZ} = \frac{AB}{XY} \qquad \Rightarrow \qquad AP = \frac{rb}{c}.$$

[13]The notation $|XYZ|$ stands for the area of $\triangle XYZ$.

Analogously, from $\triangle DPC \sim \triangle XYZ$, we have

$$\frac{|DPC|}{|XYZ|} = \frac{DC^2}{XZ^2} \quad \Rightarrow \quad |CPD| = \frac{r^2 S}{b^2}$$

and

$$\frac{PC}{YZ} = \frac{DC}{XZ} \quad \Rightarrow \quad CP = \frac{ra}{b}.$$

And from $\triangle PEF \sim \triangle XYZ$, we have

$$\frac{|PEF|}{|XYZ|} = \frac{EF^2}{YZ^2} \quad \Rightarrow \quad |EPF| = \frac{r^2 S}{a^2}$$

and

$$\frac{PE}{XY} = \frac{EF}{YZ} \quad \Rightarrow \quad EP = \frac{rc}{a}.$$

We also have $\triangle FPA \sim \triangle XYZ$, and so

$$\frac{|FPA|}{|XYZ|} = \left(\frac{PA}{YZ}\right)^2 = \left(\frac{rb}{ca}\right)^2 \quad \Rightarrow \quad |FPA| = \frac{r^2 S b^2}{c^2 a^2}.$$

Analogously, from $\triangle PCB \sim \triangle XYZ$, we have

$$\frac{|PCB|}{|XYZ|} = \left(\frac{PC}{XY}\right)^2 = \left(\frac{ra}{bc}\right)^2 \quad \Rightarrow \quad |BPC| = \frac{r^2 S a^2}{b^2 c^2}.$$

And from $\triangle EDP \sim \triangle XYZ$, we have

$$\frac{|EDP|}{|XYZ|} = \left(\frac{EP}{XZ}\right)^2 = \left(\frac{rc}{ab}\right)^2 \quad \Rightarrow \quad |DPE| = \frac{r^2 S c^2}{a^2 b^2}.$$

Since $|ABCDEF| = |APB| + |BPC| + |CPD| + |DPE| + |EPF| + |FPA|$, the given inequality we are required to show is equivalent to

$$\begin{aligned}
& |BPC| + |DPE| + |FPA| \geq |APB| + |CPD| + |EPF| \\
\Leftrightarrow \quad & \frac{r^2 S a^2}{b^2 c^2} + \frac{r^2 S c^2}{a^2 b^2} + \frac{r^2 S b^2}{c^2 a^2} \geq \frac{r^2 S}{c^2} + \frac{r^2 S}{b^2} + \frac{r^2 S}{a^2} \\
\Leftrightarrow \quad & a^4 + b^4 + c^4 \geq a^2 b^2 + c^2 a^2 + b^2 c^2 \\
\Leftrightarrow \quad & (a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 \geq 0.
\end{aligned}$$

The last inequality is trivially true, which concludes the proof. $\square$

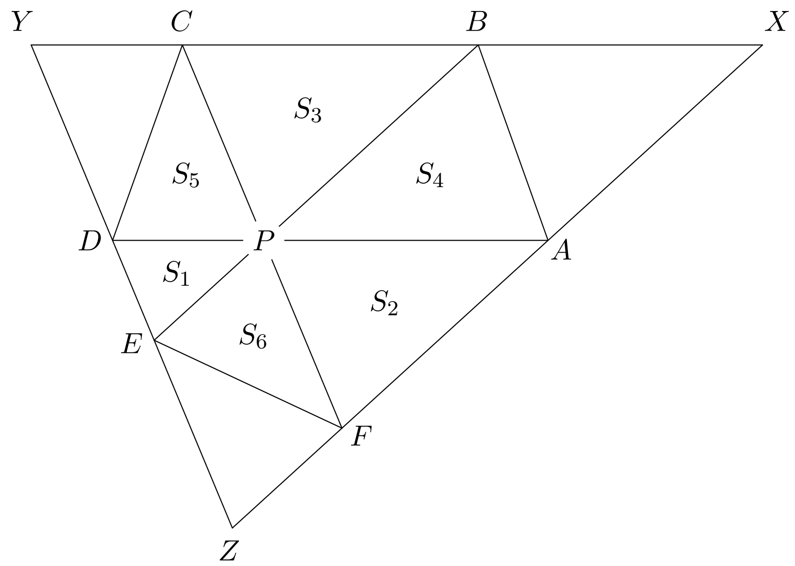
Solution 2 (Kevin Xian, year 12, James Ruse Agricultural High School, NSW)

Let X be the intersection of lines AF and BC. Let Y be the intersection of lines DE and BC. And let Z be the intersection of lines DE and AF.

Since $AB = CD$ and $ABCD$ is cyclic, it follows that $ABCD$ is an isosceles trapezium with $AD \parallel BC$. Thus $AD \parallel XY$. Similarly we have $CF \parallel YZ$ and $BE \parallel XZ$. It follows that

$$\triangle BCP \sim \triangle PDE \sim \triangle APF \sim \triangle XYZ.$$

Let $S_1 = |PDE|$, $S_2 = |PFA|$, $S_3 = |PBC|$, $S_4 = |PAB|$, $S_5 = |PCD|$, and $S_6 = |PEF|$.



Since $BE \parallel XZ$ and $AD \parallel XY$, it follows that $APBX$ is a parallelogram. Hence $|XAB| = |PAB| = S_4$. Similarly we have $|YCD| = S_5$ and $|ZEF| = S_6$. The inequality we are asked to prove is

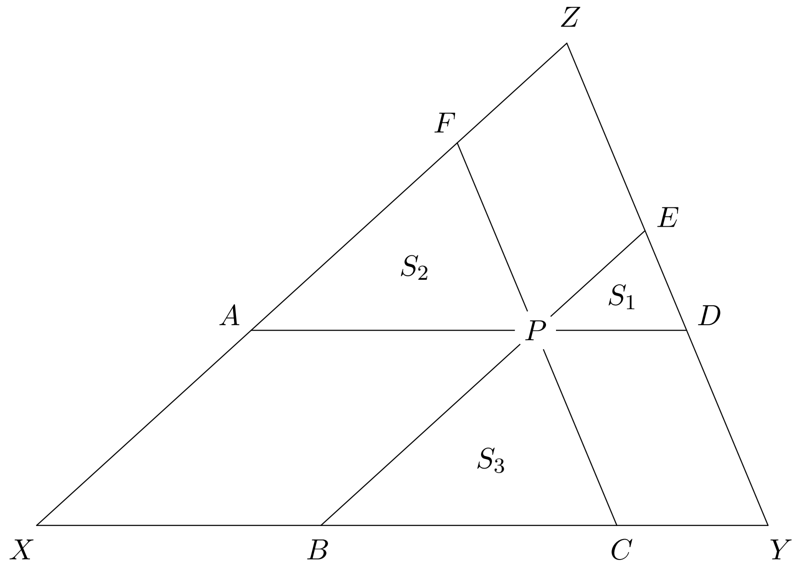
$$\begin{aligned}
& S_1 + S_2 + S_3 + S_4 + S_5 + S_6 \geq 2(S_4 + S_5 + S_6) \\
\Leftrightarrow \quad & S_1 + S_2 + S_3 \geq S_4 + S_5 + S_6 \\
\Leftrightarrow \quad & 3(S_1 + S_2 + S_3) \geq 2(S_4 + S_5 + S_6) + S_1 + S_2 + S_3 \\
\Leftrightarrow \quad & 3(S_1 + S_2 + S_3) \geq |XYZ| \\
\Leftrightarrow \quad & \frac{S_1 + S_2 + S_3}{|XYZ|} \geq \frac{1}{3}.
\end{aligned}$$

For the sake of less clutter, we draw another diagram. The essential details are that $AD \parallel XY$, $BE \parallel XZ$, and $CF \parallel YZ$, and that AD, BE, and CF are concurrent at P.[14]

[14] It turns out that if this is the case, then the inequality

$$\frac{S_1 + S_2 + S_3}{|XYZ|} \geq \frac{1}{3}$$

is true in general. In particular, as shall be seen, it is true whether or not we have $AB = CD = EF$.



Let $u = XB$, $v = BC$, and $w = CY$. Then since $\triangle PDE$, $\triangle APF$, and $\triangle BCP$ are all similar to $\triangle XYZ$, we have

$$\frac{S_1}{|XYZ|} = \left(\frac{PD}{XY}\right)^2 = \left(\frac{CY}{XY}\right)^2 = \frac{w^2}{(u+v+w)^2},$$

$$\frac{S_2}{|XYZ|} = \left(\frac{AP}{XY}\right)^2 = \left(\frac{XB}{XY}\right)^2 = \frac{u^2}{(u+v+w)^2},$$

and

$$\frac{S_3}{|XYZ|} = \left(\frac{BC}{XY}\right)^2 = \frac{v^2}{(u+v+w)^2}.$$

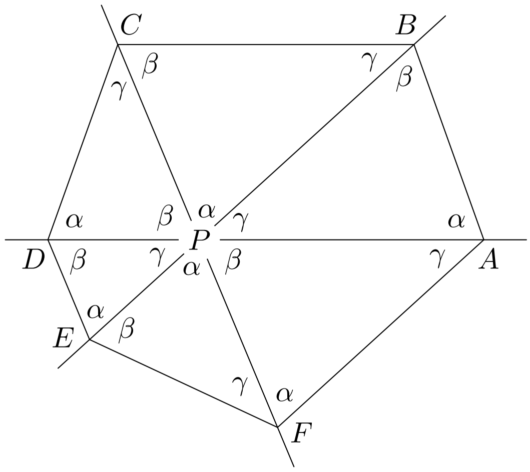
Thus the inequality is equivalent to

$$\begin{aligned} & \frac{u^2+v^2+w^2}{(u+v+w)^2} \geq \frac{1}{3} \\ \Leftrightarrow \quad & u^2+v^2+w^2 \geq uv+vw+wu \\ \Leftrightarrow \quad & (u-v)^2+(v-w)^2+(w-u)^2 \geq 0 \end{aligned}$$

The last inequality is trivially true, which concludes the proof. $\square$

Solution 3 (Matthew Cheah, year 11, Penleigh and Essendon Grammar School, VIC)

We define α, β, and γ, and angle chase the diagram as was done in solution 1 to part (b).



The inequality we are required to prove equivalent to

$$|BPC| + |DPE| + |FPA| \geq |APB| + |CPD| + |EPF|.$$

We can scale the diagram so that $AB = CD = EF = 1$. We may compute

$$\begin{aligned} PA &= \frac{\sin\beta}{\sin\gamma}, \quad PB = \frac{\sin\alpha}{\sin\gamma} \quad \text{(sine rule } \triangle APB\text{)} \\ PC &= \frac{\sin\alpha}{\sin\beta}, \quad PD = \frac{\sin\gamma}{\sin\beta} \quad \text{(sine rule } \triangle CPD\text{)} \\ PE &= \frac{\sin\gamma}{\sin\alpha}, \quad PF = \frac{\sin\beta}{\sin\alpha} \quad \text{(sine rule } \triangle EPF\text{)}. \end{aligned}$$

Using the $\frac{1}{2}bc\sin\alpha$ formula for the area of a triangle we compute

$$\begin{aligned} |APB| &= \frac{\sin\alpha\sin\beta}{2\sin\gamma}, \qquad |CPD| = \frac{\sin\alpha\sin\gamma}{2\sin\beta}, \qquad |EPF| = \frac{\sin\beta\sin\gamma}{2\sin\alpha} \\ |BPC| &= \frac{\sin^3\alpha}{2\sin\beta\sin\gamma}, \quad |DPE| = \frac{\sin^3\gamma}{2\sin\alpha\sin\beta}, \quad |FPA| = \frac{\sin^3\beta}{2\sin\alpha\sin\gamma}. \end{aligned}$$

Thus the inequality is equivalent to

$$\frac{\sin^3\alpha}{2\sin\beta\sin\gamma} + \frac{\sin^3\beta}{2\sin\alpha\sin\gamma} + \frac{\sin^3\gamma}{2\sin\alpha\sin\beta} \geq \frac{\sin\beta\sin\gamma}{2\sin\alpha} + \frac{\sin\alpha\sin\gamma}{2\sin\beta} + \frac{\sin\alpha\sin\beta}{2\sin\gamma}.$$

After clearing denominators, this last inequality becomes

$$\sin^4\alpha + \sin^4\beta + \sin^4\gamma \geq \sin^2\alpha\sin^2\beta + \sin^2\beta\sin^2\gamma + \sin^2\alpha\sin^2\gamma.$$

But this can be rewritten as

$$(\sin^2\alpha - \sin^2\beta)^2 + (\sin^2\beta - \sin^2\gamma)^2 + (\sin^2\gamma - \sin^2\alpha)^2 \geq 0,$$

which is obviously true. ☐

Score Distribution/Problem

Number of Students/Score	Problem Number							
	1	2	3	4	5	6	7	8
0	16	22	32	65	12	28	28	88
1	4	3	7	6	14	14	47	0
2	6	0	3	2	13	10	1	1
3	18	1	12	1	9	4	0	0
4	0	0	3	3	1	4	0	2
5	0	1	3	0	5	1	1	0
6	2	6	11	6	24	6	0	0
7	51	64	26	14	19	30	20	6
Average	4.5	5.1	3.3	1.6	3.8	3.2	2.0	0.5

AUSTRALIAN MATHEMATICAL OLYMPIAD RESULTS

Name	School	Year
Gold and Perfect Score		
Matthew Cheah	Penleigh and Essendon Grammar School VIC	11
Seyoon Ragavan	Knox Grammar School NSW	12
Wilson Zhao	Killarra High School NSW	12
Gold		
Jongmin Lim	Killarra High School NSW	12
Yong See Foo	Nossal High School VIC	12
Kevin Xian	James Ruse Agricultural High School NSW	12
Michelle Chen	Methodist Ladies College VIC	12
Jerry Mao	Caulfield Grammar School VIC	10
William Hu	Christ Church Grammar School WA	10
Ilia Kucherov	Westall Secondary College VIC	12
Silver		
Guowen Zhang	St Joseph's College QLD	10
Charles Li	Camberwell Grammar School VIC	10
Jack Liu	Brighton Grammar School VIC	10
Tony Jiang	Scotch College VIC	11
James Bang	Baulkham Hills High School NSW	9
Yiannis Fam	Wellington College NZ	12*
Leo Li	Christ Church Grammar School WA	12
Miles Lee	Auckland International College NZ	13*
Kevin Shen	Saint Kentigern College NZ	13*
Linus Cooper	James Ruse Agricultural High School NSW	10
Thomas Baker	Scotch College VIC	12
Eric Sheng	Newington College NSW	12
Keiran Lewellen	Te Kura (The Correspondence School) NZ	11*
Barnard Patel	Wellington College NZ	12*
Michael Robertson	Dickson College ACT	12
Bronze		
Sharvil Kesawani	Merewether High School NSW	9
Jeff (Zefeng) Li	Camberwell Grammar School VIC	9

Name	School	Year
Harish Suresh	James Ruse Agricultural High School NSW	12
Anthony Pisani	St Paul's Anglican Grammar School VIC	9
Anthony Tew	Pembroke School SA	10
Elliot Murphy	Canberra Grammar School ACT	11
Zoe Schwerkolt	Fintona Girls' School VIC	12
Yuting Niu	Auckland International College NZ	12*
Chuanye (Andrew) Chen	Auckland Grammar School NZ	11*
Isabel Longbottom	Rossmoyne Senior High School WA	11
Steven Lim	Hurlstone Agricultural High School NSW	10
Yuelin Shen	Scotch College WA	11
William Song	Scotch College VIC	12
Xutong Wang	Auckland International College NZ	10*
Yasiru Jayasooriya	James Ruse Agricultural High School NSW	8
Hadyn Tang	Trinity Grammar School VIC	7
Andrew Chen	Saint Kentigern College NZ	11*
Austin Zhang	Sydney Grammar School NSW	11
Daniel Jones	All Saints Anglican School QLD	11
Hang Sheng	Rossmoyne Senior High School WA	12
Stacey Tian	St Cuthbert's College NZ	12*
Anand Bharadwaj	Trinity Grammar School VIC	10
Bobby Dey	James Ruse Agricultural High School NSW	11
Shivasankaran Jayabalan	Rossmoyne Senior High School WA	10
Jodie Lee	Seymour College SA	11
William Li	Barker College NSW	10
Alex Lugovskoy	Willetton Senior High School WA	12
Forbes Mailler	Canberra Grammar School ACT	10
Virinchi Rallabhandi	Perth Modern School WA	12
Dibyendu Roy	Sydney Boys High School NSW	11

* indicates New Zealand school year