AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

A DEPARTMENT OF THE AUSTRALIAN MATHEMATICS TRUST

2017 AUSTRALIAN MATHEMATICAL OLYMPIAD

DAY 1
Tuesday, 14 February 2017
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. For which integers $n \geq 2$ is it possible to write the numbers $1, 2, 3, \ldots, n$ in a row in some order so that any two numbers written next to each other in the row differ by 2 or 3?

2. Given five distinct integers, consider the ten differences formed by pairs of these numbers. (Note that some of these differences may be equal.)

   Determine the largest integer that is certain to divide the product of these ten differences, regardless of which five integers were originally given.

3. Determine all functions f defined for real numbers and taking real numbers as values such that
$$f(x^2 + f(y)) = f(xy)$$
   for all real numbers x and y.

4. Suppose that S is a set of 2017 points in the plane that are not all collinear.

   Prove that S contains three points that form a triangle whose circumcentre is not a point in S.

AUSTRALIAN MATHEMATICS TRUST

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2017 AUSTRALIAN MATHEMATICAL OLYMPIAD

DAY 2
Wednesday, 15 February 2017
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. Determine the number of positive integers n less than $1\,000\,000$ for which the sum

$$\frac{1}{2\times\lfloor\sqrt{1}\rfloor+1}+\frac{1}{2\times\lfloor\sqrt{2}\rfloor+1}+\frac{1}{2\times\lfloor\sqrt{3}\rfloor+1}+\cdots+\frac{1}{2\times\lfloor\sqrt{n}\rfloor+1}$$

is an integer.

(Note that $\lfloor x\rfloor$ denotes the largest integer that is less than or equal to x.)

6. The circles K_1 and K_2 intersect at two distinct points A and M. Let the tangent to K_1 at A meet K_2 again at B, and let the tangent to K_2 at A meet K_1 again at D. Let C be the point such that M is the midpoint of AC.

Prove that the quadrilateral $ABCD$ is cyclic.

7. There are 1000 athletes standing equally spaced around a circular track of length 1 kilometre.

(a) How many ways are there to divide the athletes into 500 pairs such that the two members of each pair are 335 metres apart around the track?

(b) How many ways are there to divide the athletes into 500 pairs such that the two members of each pair are 336 metres apart around the track?

8. Let $f(x) = x^2 - 45x + 2$.

Find all integers $n \geq 2$ such that exactly one of the numbers

$$f(1), f(2), \ldots, f(n)$$

is divisible by n.

1. **Comment** This problem was solved by 80 of the 104 contestants. We present some of the many different approaches that were possible in arriving at a complete solution.

Solution 1

Answer: All integers $n \geq 4$.

Clearly $n = 2$ does not work because 1 and 2 differ by 1. Also $n = 3$ does not work because 2 cannot go next to either 1 or 3. It remains to show that all $n \geq 4$ work.

Using 2,4,1,3 as a starting point, at each step we simply write the next number on either side of the list with the odd numbers on the left and the even numbers on the right until we reach the number n as follows.

$$\ldots, 9, 7, 5, \underbracket{2, 4, 1, 3}, 6, 8, 10, \ldots. \qquad \square$$

Comments The above solution seems to be the simplest. This was the most frequent solution found by the contestants.

In the remaining solutions, we only demonstrate that all $n \geq 4$ work.

Solution 2

For $n = 4$ and 5, we have working sequences 2,4,1,3 and 3,5,2,4,1, respectively.

For $n \geq 6$ we have the following two cases.

Case 1 n is even

We have the working sequence

$$\underbrace{1, 3, 5, \ldots, n-3, n-5}_{\text{area 1}}, n-2, n, n-3, n-1, \underbrace{n-4, n-6, \ldots, 6, 4, 2}_{\text{area 2}}.$$

Here area 1 consists of the odd numbers from 1 up to $n-5$, while area 2 consists of the even numbers from $n-4$ down to 2.

Case 2 n is odd

We have the working sequence

$$\underbrace{2, 4, 6, \ldots, n-3, n-5}_{\text{area 1}}, n-2, n, n-3, n-1, \underbrace{n-4, n-6, \ldots, 5, 3, 1}_{\text{area 2}}.$$

Here area 1 consists of the even numbers from 2 up to $n-5$, while area 2 consists of the odd numbers from $n-4$ down to 1. $\square$

Comment This solution is closely related to the first solution. Can you explain the connection?

Solution 3

For $n = 4$ and 5, we have working sequences 2,4,1,3 and 3,5,2,4,1, respectively.

In each of the three cases that follow, k is any integer greater than or equal to 2.

Case 1 $n = 3k$

We have the working sequence

$$1, \underbrace{3, 6, 9, \ldots, 3k}_{\text{area 1}}, \underbrace{3k-2, 3k-5, 3k-8, \ldots, 4}_{\text{area 2}}, \underbrace{2, 5, 8, \ldots, 3k-1}_{\text{area 3}}.$$

Here the numbers in area 1 go up by 3 each time, the numbers in area 2 go down by 3 each time, and the numbers in area 3 go up by 3 each time.

Case 2 $n = 3k+1$

We simply append $3k+1$ to the above working sequence for $n = 3k$ as shown below.

$$1, \underbrace{3, 6, 9, \ldots, 3k}_{\text{area 1}}, \underbrace{3k-2, 3k-5, 3k-8, \ldots, 4}_{\text{area 2}}, \underbrace{2, 5, 8, \ldots, 3k-1}_{\text{area 3}}, 3k+1$$

Case 3 $n = 3k+2$

We have the working sequence

$$1, \underbrace{3, 6, 9, \ldots, 3k}_{\text{area 1}}, \underbrace{3k+2, 3k-1, 3k-4, \ldots, 2}_{\text{area 2}}, \underbrace{4, 7, 10, \ldots, 3k+1}_{\text{area 3}}.$$

Again the numbers in area 1 go up by 3 each time, the numbers in area 2 go down by 3 each time, and the numbers in area 3 go up by 3 each time. $\square$

Solution 4

Consider the following repeating pattern of groups of four numbers.

$$\underbrace{2, 4, 1, 3}_{\text{group 1}}, \underbrace{6, 8, 5, 7}_{\text{group 2}}, \underbrace{10, 12, 9, 11}_{\text{group 3}}, \underbrace{14, 16, 13, 15}_{\text{group 4}}, \ldots \tag{1}$$

Note that each group of four is formed by adding 4 to each member of the previous group of four. In each of the four cases that follow, k is any positive integer.

Case 1 $n = 4k$

The first k groups in (1) form a working sequence.

Case 2 $n = 4k + 1$

Take the first k groups in (1) to get a working sequence for $n = 4k$ whose last member is $4k - 1$. Appending $4k + 1$ at the end yields a working sequence for $n = 4k + 1$.

Case 3 $n = 4k + 2$

Take the first $k - 1$ groups in (1) to get a working sequence for $n = 4k - 4$ whose last member is $4k - 5$. Appending

$$4k - 3, 4k - 1, 4k + 1, 4k - 2, 4k, 4k + 2$$

at the end yields a working sequence for $n = 4k + 2$.

Case 4 $n = 4k + 3$

Take the first $k - 1$ groups in (1) to get a working sequence for $n = 4k - 4$ whose last member is $4k - 5$. Appending

$$4k - 3, 4k - 1, 4k + 2, 4k, 4k - 2, 4k + 1, 4k + 3$$

at the end yields a working sequence for $n = 4k + 3$. $\square$

Solution 5

We say that a working sequence for n is *helpful* if its rightmost term is $n-1$. We shall prove that there is a helpful working sequence for each integer $n \geq 4$.

To start with, we have the following helpful working sequences for $n = 4, 5, 6, 7, 8$.

$$\begin{aligned}
n = 4 :&\quad 2, 4, 1, 3 \\
n = 5 :&\quad 1, 3, 5, 2, 4 \\
n = 6 :&\quad 1, 3, 6, 4, 2, 5 \\
n = 7 :&\quad 2, 5, 7, 4, 1, 3, 6 \\
n = 8 :&\quad 1, 3, 6, 8, 5, 2, 4, 7
\end{aligned}$$

Moreover, we can transform any helpful working sequence for n into a helpful working sequence for $n+5$ by appending

$$n+1, n+3, n+5, n+2, n+4$$

at the end.

The result now follows. $\square$

Solution 6

Call a working sequence for n *useful* if $n-2$ and n are adjacent terms of the sequence. We shall prove that there is a useful working sequence for each integer $n \geq 4$.

To start with, we have the following useful working sequences for $n = 4, 5, 6, 7, 8$.

$$\begin{aligned}
n = 4 : &\quad \boxed{2, 4}, 1, 3 \\
n = 5 : &\quad 1, \boxed{3, 5}, 2, 4 \\
n = 6 : &\quad 1, 3, \boxed{6, 4}, 2, 5 \\
n = 7 : &\quad 2, \boxed{5, 7}, 4, 1, 3, 6 \\
n = 8 : &\quad 2, 4, 7, 5, \boxed{8, 6}, 3, 1
\end{aligned}$$

Moreover, we can transform any useful working sequence for n into a useful working sequence for $n + 5$ by inserting

$$n + 1, n + 4, n + 2, \boxed{n + 5, n + 3}$$

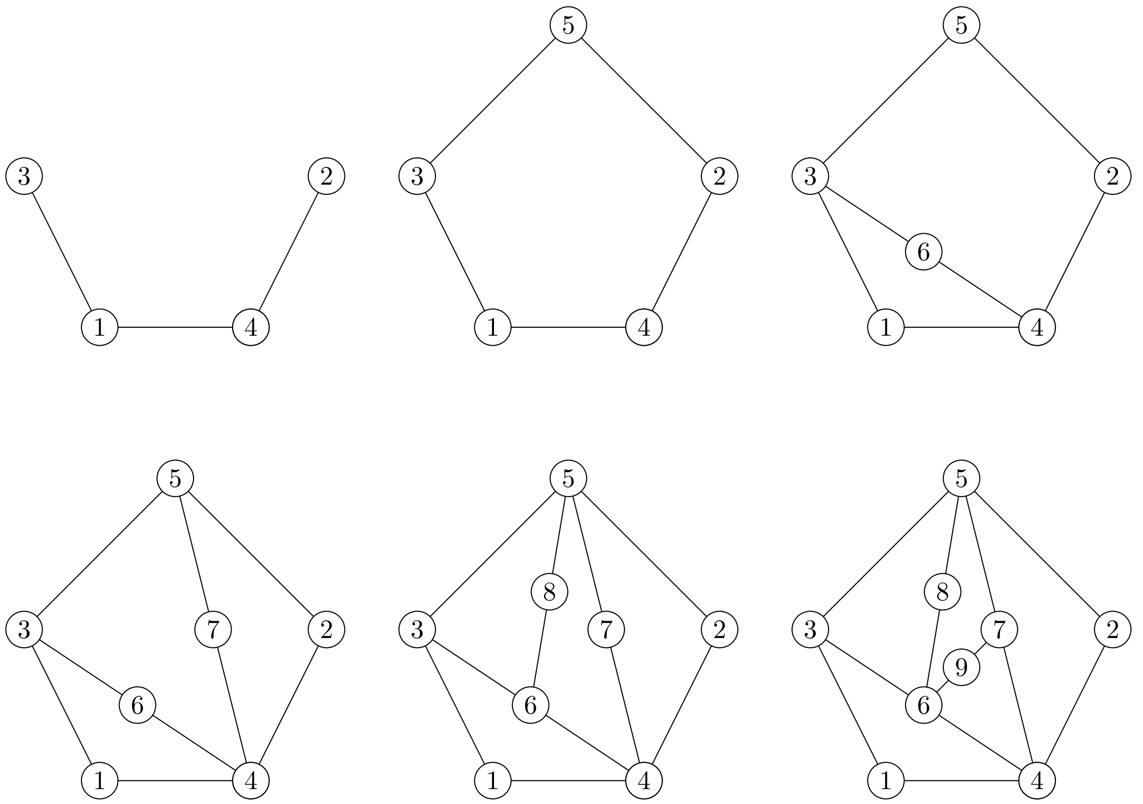
or its reverse in between $n - 2$ and n.

The result now follows. $\square$

Comment 1 We invite the reader to find answers to the following variations on the given problem.

- The same problem with the additional requirement that the sequence starts with 1.[1]
- The same problem with the additional requirements that the sequence starts with 1 and ends with n.
- The same problem with the additional requirement that the sequence is circular, that is, the difference between the first and last term is also 2 or 3.

Comment 2 It is convenient to visualise the situation using a graph G. Each of the positive integers $1, 2, \ldots, n$ corresponds to a vertex of G, and two vertices are connected by an edge if and only if they differ by two or three. It is fairly simple to draw the graph by adding one vertex at a time as depicted below for $n = 4, 5, 6, 7, 8, 9$.



Finding a valid sequence is equivalent to finding a path in the corresponding graph that visits each vertex exactly once.

[1]Actually this was already done in solution 3.

2. **Solution**

Answer: 288

For the integers 1, 2, 3, 4, 5, we calculate directly that the product of the ten differences is $2^5 \times 3^2 = 288$. Hence the answer is a factor of 288.

To show that the answer is 288, we shall show that 2^5 and 3^2 divide the product, P say, of the ten differences.

Part 1 Show that 2^5 divides P.

By the pigeonhole principle, at least three of the five numbers have the same parity.

If four or more of the five numbers have the same parity, then each of the $\binom{4}{2} = 6$ differences between them is even. Thus 2^6, and hence 2^5 divides P.

If, on the other hand, exactly three of the numbers, say a, b, and c are of one parity, and the other two, say d and e are of the other parity, then each of the differences $a - b$, $a - c$, $b - c$, and $d - e$ are even. Since a, b, and c have the same parity, either

$$a, b, c \in \{0, 2\} \pmod 4 \quad \text{or} \quad a, b, c \in \{1, 3\} \pmod 4.$$

In each case at least one of the differences $a - b$, $a - c$, or $b - c$ is divisible by 4. Hence again 2^5 divides P.

Part 2 Show that 3^2 divides P.

By the pigeonhole principle, two of the five numbers are congruent modulo 3.

If three of the numbers are congruent modulo 3, then each of the $\binom{3}{2} = 3$ differences between them is divisible by 3. Thus 3^3, and hence 3^2 divides P.

If at most two numbers are congruent to each other modulo 3, say $a \equiv b \pmod 3$, then the other three numbers must come from the other two congruence classes. Again by the pigeohole principle, it follows that two of the other three numbers are congruent modulo 3, say $c \equiv d \pmod 3$. It follows that 3^2 divides P. $\square$

3. **Solution 1**

Answers: $f(x) = c$ for any real constant c.

For reference we are given

$$f(x^2 + f(y)) = f(xy) \quad \text{for all } x, y \in \mathbb{R}. \tag{1}$$

Put $y = 0$ in (1) to find

$$f(x^2 + f(0)) = f(0) \quad \text{for all } x \in \mathbb{R}. \tag{2}$$

Observe that $x^2 + f(0)$ covers all real numbers that are greater than or equal to $f(0)$. It follows that

$$f(x) = f(0) \quad \text{whenever } x \geq f(0). \tag{3}$$

Next choose $a \geq f(0)$ with $a \neq 0$. Put $y = a$ into (1) and use (3) and then (2) to find

$$f(xa) = f(x^2 + f(a)) = f(x^2 + f(0)) = f(0).$$

Since $a \neq 0$, the expression xa covers all real numbers as x ranges over the reals. Hence $f(x) = f(0)$ for all $x \in \mathbb{R}$. Thus f is a constant function. It is readily seen that any such function satisfies (1). $\square$

Solution 2

For reference we are given

$$f(x^2 + f(y)) = f(xy) \quad \text{for all } x, y \in \mathbb{R}. \tag{1}$$

Put $y = 0$ in (1) to deduce

$$f(x^2 + f(0)) = f(0) \quad \text{for all } x \in \mathbb{R}. \tag{2}$$

Put $x = 0$ in (1) to deduce

$$f(f(y)) = f(0) \quad \text{for all } y \in \mathbb{R}. \tag{3}$$

Replacing y with $f(y)$ in (1), and using (3) and then (2), we have for all $x, y \in \mathbb{R}$

$$f(xf(y)) = f(x^2 + f(f(y)) = f(x^2 + f(0)) = f(0).$$

If $f(y) = 0$ for all $y \in \mathbb{R}$, then it is readily checked that this function satisfies (1).

If, on the other hand, there is a real number y such that $f(y) \neq 0$, then $xf(y)$ covers all real numbers as x ranges over the reals. Hence $f(x) = f(0)$ for all $x \in \mathbb{R}$. So f is a constant function. It is readily seen that any such function satisfies (1). $\square$

4. **Solution 1**

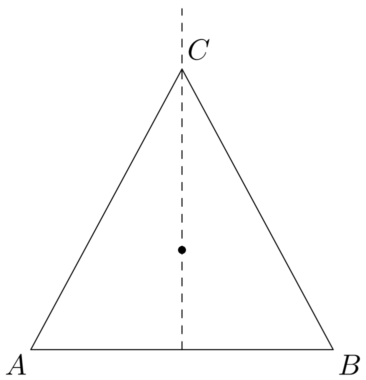
Let A and B be two points of S whose distance apart is minimal. Let ℓ be the perpendicular bisector of AB.

Case 1 The line ℓ contains no point of S.

Since not all points of S are collinear, there exists a point X not on the line AB. Consequently, the circumcentre of $\triangle ABX$, which lies on ℓ, is not in S, as required.

Case 2 The line ℓ contains at least one point of S.

Let C be a point in S, lying on ℓ, and of minimal distance to the line AB.



Recall AB is the minimal distance between points of S. Thus $CA = CB \geq AB$. Therefore $\angle ACB$ is the smallest angle in $\triangle ABC$, and so $\angle ACB \leq 60°$. Since $\triangle ABC$ is isosceles with $CA = CB$, we also have $\angle BAC = \angle CBA < 90°$. Hence $\triangle ABC$ is acute.

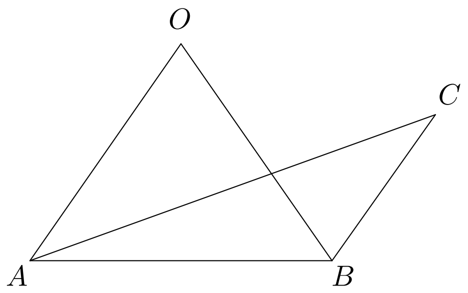
Acute triangles contain their circumcentres, so the circumcentre of $\triangle ABC$, which lies on ℓ, also lies inside $\triangle ABC$ and is therefore closer to the line AB than the point C. Since no point in S, that is also on ℓ, lies closer to AB than C, it follows that the circumcentre of $\triangle ABC$ is not in S, as required. $\square$

Solution 2

Let A and B be any two consecutive points on the convex hull of S. Orient the diagram so that AB is horizontal and no point of S lies below the line AB. Of all points in S that lie strictly above the line AB, let C be a point such that $\angle ACB$ is maximal. We shall prove that the circumcentre, O say, of $\triangle ABC$ is not in S.

Case 1 $0^\circ < \angle ACB < 90^\circ$

We have $\angle AOB = 2\angle ACB$. Hence $0^\circ < \angle ACB < \angle AOB < 180^\circ$. Thus O lies above the line AB and satisfies $\angle AOB > \angle ACB$. From the maximality of $\angle ACB$, we conclude that O is not in S, as desired.



Case 2 $\angle ACB = 90^\circ$

The point O is the midpoint of AB. Since A and B were chosen to be consecutive points on the convex hull of S, it follows that O is not in S, as desired.

Case 3 $90^\circ < \angle ACB < 180^\circ$

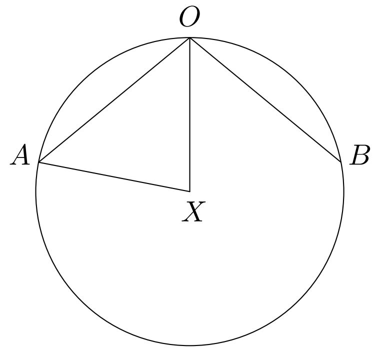
The point O lies below the line AB and so is not in S, as desired. $\square$

Solution 3

Let ABC be a triangle formed from points in S that has minimal circumradius, R say, and let O be the circumcentre of $\triangle ABC$.

If O is not in S, then we are done.

If O is in S, then since the three directed angles (modulo 360°) $\angle AOB$, $\angle BOC$, and $\angle COA$ sum to 360°, we may suppose without loss of generality that $\angle AOB \leq 120°$. Let X be the circumcentre of $\triangle AOB$. To complete the proof we shall show that X is not in S.



By symmetry OX bisects $\angle AOB$. Hence $\angle XAO = \angle AOX = \frac{1}{2}\angle AOB \leq 60°$.

If $\angle XAO = \angle AOX < 60°$, then $\angle OXA > 60°$ is strictly the largest angle in $\triangle AOX$. Thus OA is strictly the largest side of $\triangle AOX$. So $OX < OA$. Hence $\triangle AOB$ has circumradius $OX < OA = R$, which contradicts the minimality of R.

If, on the other hand, $\angle XAO = \angle AOX = 60°$, then $\triangle AOX$ is equilateral and has circumradius equal to $OA/\sqrt{3} < OA = R$. Since $\triangle ABC$ has minimal circumradius formed from points in S, it follows that X is not in S, as desired. $\square$

Solution 4

Let ABC be a triangle formed from points in S that has minimal circumradius, and let O be its circumcentre. For any triangle UVW, let R_{UVW} denote its circumradius.

If O is not in S, then we are done.

If O is in S, let α, β, and γ be the angles at A, B, and C, respectively, in $\triangle ABC$.

Case 1 $\triangle ABC$ is acute.

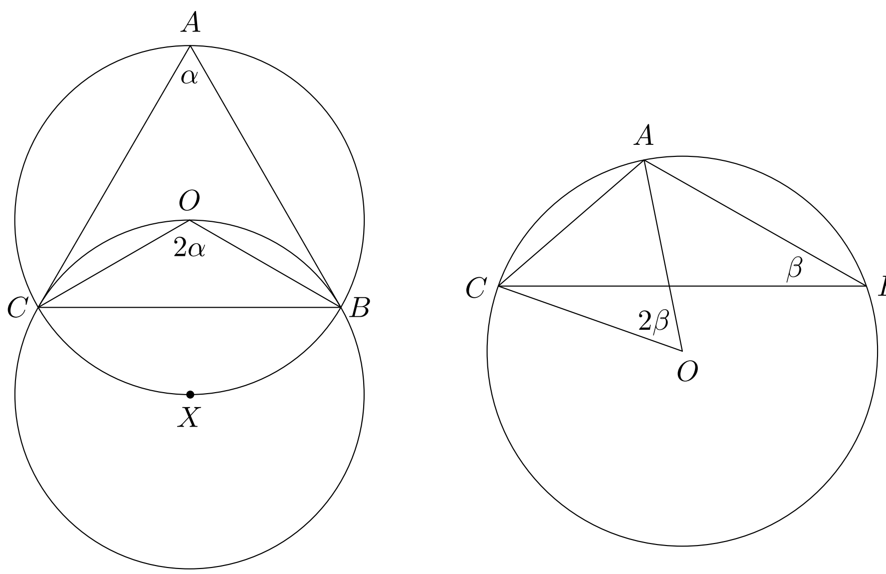
Note that $\angle BOC = 2\alpha$. By the minimality of R_{ABC}, we have $R_{ABC} \leq R_{BOC}$. Using the extended sine rule in $\triangle ABC$ and $\triangle BOC$, to compute these circumradii, we have

$$\frac{BC}{\sin\alpha} \leq \frac{BC}{\sin 2\alpha} = \frac{BC}{2\sin\alpha\cos\alpha} \quad \Rightarrow \quad \cos\alpha \leq \frac{1}{2}.$$

Since $0° < \alpha < 90°$ it follows that $\alpha \geq 60°$. Similarly, $\beta \geq 60°$ and $\gamma \geq 60°$. But $\alpha + \beta + \gamma = 180°$. Hence $\alpha = \beta = \gamma = 60°$, and so $\triangle ABC$ is equilateral.

Let X be the circumcentre of $\triangle BOC$. If X is not in S, then we are done.

If X is in S, then $\angle OXC = 2\angle OBC = 60°$, and $XO = XC$, so that $\triangle OXC$ is equilateral. However, equilateral $\triangle OXC$ is clearly smaller than equilateral $\triangle ABC$. Thus $R_{OXC} < R_{ABC}$, which contradicts the minimality of R_{ABC}.



Case 2 $\triangle ABC$ is not acute.

Without loss of generality we may suppose that $\alpha \geq 90°$. It follows that $\beta + \gamma \leq 90°$. Without loss of generality we may assume that $\angle\beta \leq 45°$.

Note that $\angle AOC = 2\beta \leq 90°$. By the minimality of R_{ABC}, we have $R_{ABC} \leq R_{AOC}$. Using the extended sine rule in $\triangle ABC$ and $\triangle AOC$, to compute these circumradii, we have

$$\frac{AC}{\sin\beta} \leq \frac{AC}{\sin 2\beta} \quad \Rightarrow \quad \sin 2\beta \leq \sin\beta.$$

However this is a contradiction because the sine function is strictly increasing on the interval $[0, 90°]$. Hence this case does not occur. $\square$

Solution 5 (Ivan Guo, AMOC Senior Problems Committee)

We will prove a stronger statement, namely, that S contains three non-collinear points whose circumcircle does not contain a point of S *anywhere* in its interior.

Let A and B be two points of S separated by minimal distance. Consider the circle with diameter AB. Note that this circle does not contain any other points of S.

One side of the line AB contains at least one point of S. Continuously expand the circle towards that side, while making sure that it still passes through A and B. Eventually, it must hit a third point of S. At this stage, the circle meets (at least) three points of S but contains no points of S in its interior. $\square$

Comment (Adrian Agisilaou, AMO marker)

Let S be any set of points that are not all collinear, and let H be the convex hull of S. It is always possible to find a triangulation[4] of H with the property that the circumcircle of each such triangle does not contain any points of S strictly in its interior. Such a triangulation is called a *Delaunay triangulation*. See https://en.wikipedia.org/wiki/Delaunay_triangulation for more details. Such triangulations are a topic in contemporary mathematical research.

[4]By a *triangulation*, we mean a partition of H into triangles so that for each such triangle T, its three vertices are points of S and no other point on the perimeter of T is a point of S.

5. **Solution 1**

Answer: 999

For reference the given sum is

$$S=\frac{1}{2\left\lfloor\sqrt{1}\right\rfloor+1}+\frac{1}{2\left\lfloor\sqrt{2}\right\rfloor+1}+\cdots+\frac{1}{2\left\lfloor\sqrt{n}\right\rfloor+1}.$$

Let m be the unique positive integer satisfying $m^2 \le n < (m+1)^2$.

For each positive integer r, consider how many positive integers x there are such that $\left\lfloor\sqrt{x}\right\rfloor = r$. We require $r^2 \le x < (r+1)^2$. So there are exactly $(r+1)^2 - r^2 = 2r+1$ values of x with $\left\lfloor\sqrt{x}\right\rfloor = r$. It follows that

$$\frac{1}{2\left\lfloor\sqrt{r^2}\right\rfloor+1}+\frac{1}{2\left\lfloor\sqrt{r^2+1}\right\rfloor+1}+\cdots+\frac{1}{2\left\lfloor\sqrt{(r+1)^2-1}\right\rfloor+1}=(2r+1)\times\frac{1}{2r+1}=1. \quad (1)$$

Using the above results, we can split S up into the following m smaller sums.

$$\begin{aligned}
S=&\frac{1}{2\left\lfloor\sqrt{1^2}\right\rfloor+1}+\frac{1}{2\left\lfloor\sqrt{1^2+1}\right\rfloor+1}+\frac{1}{2\left\lfloor\sqrt{2^2-1}\right\rfloor+1}\\
&+\frac{1}{2\left\lfloor\sqrt{2^2}\right\rfloor+1}+\frac{1}{2\left\lfloor\sqrt{2^2+1}\right\rfloor+1}+\cdots+\frac{1}{2\left\lfloor\sqrt{3^2-1}\right\rfloor+1}\\
&+\frac{1}{2\left\lfloor\sqrt{3^2}\right\rfloor+1}+\frac{1}{2\left\lfloor\sqrt{3^2+1}\right\rfloor+1}+\cdots+\frac{1}{2\left\lfloor\sqrt{4^2-1}\right\rfloor+1}\\
&\vdots\\
&+\frac{1}{2\left\lfloor\sqrt{(m-1)^2}\right\rfloor+1}+\frac{1}{2\left\lfloor\sqrt{(m-1)^2+1}\right\rfloor+1}+\cdots+\frac{1}{2\left\lfloor\sqrt{m^2-1}\right\rfloor+1}\\
&+\frac{1}{2\left\lfloor\sqrt{m^2}\right\rfloor+1}+\frac{1}{2\left\lfloor\sqrt{m^2+1}\right\rfloor+1}+\cdots+\frac{1}{2\left\lfloor\sqrt{n}\right\rfloor+1}.
\end{aligned}$$

From (1), each of the first $m-1$ lines in the above has sum equal to 1. From the discussion immediately preceding (1), the last line has sum equal to

$$\underbrace{\frac{1}{2m+1}+\frac{1}{2m+1}+\cdots+\frac{1}{2m+1}}_{n-m^2+1\text{ terms}}=\frac{n-m^2+1}{2m+1}.$$

However, since $m^2 \le n < (m+1)^2$ we see that

$$0<\frac{n-m^2+1}{2m+1}\le 1.$$

Hence S is an integer if and only if $n-m^2+1=2m+1$, that is, $n=(m+1)^2-1$.

Since n is a positive integer less than one million, the sum S is an integer precisely when $n = 2^2-1, 3^2-1, \ldots, 1000^2-1$. Thus there are exactly 999 values of n for which S is an integer. $\square$

Solution 2

For each positive integer n let

$$f(n) = \frac{1}{2\left\lfloor\sqrt{1}\right\rfloor + 1} + \frac{1}{2\left\lfloor\sqrt{2}\right\rfloor + 1} + \cdots + \frac{1}{2\left\lfloor\sqrt{n}\right\rfloor + 1}.$$

Note that f is strictly increasing.

We claim that $f(k^2 - 1) = k - 1$ for each integer $k \geq 2$. We prove this by induction.

For the base case $k = 2$, we calculate

$$f(3) = \frac{1}{2\left\lfloor\sqrt{1}\right\rfloor + 1} + \frac{1}{2\left\lfloor\sqrt{2}\right\rfloor + 1} + \frac{1}{2\left\lfloor\sqrt{3}\right\rfloor + 1} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.$$

Hence the base case checks out.

For the inductive step, let us assume that $f(k^2 - 1) = k - 1$ for some integer $k \geq 2$. We calculate

$$\begin{aligned} f((k+1)^2 - 1) &= f(k^2 - 1) + \frac{1}{2\left\lfloor\sqrt{k^2}\right\rfloor + 1} + \frac{1}{2\left\lfloor\sqrt{k^2+1}\right\rfloor + 1} + \cdots + \frac{1}{2\left\lfloor\sqrt{k^2+2k}\right\rfloor + 1} \\ &= k - 1 + \underbrace{\frac{1}{2k+1} + \frac{1}{2k+1} + \cdots + \frac{1}{2k+1}}_{2k+1 \text{ terms}} \\ &= k. \end{aligned}$$

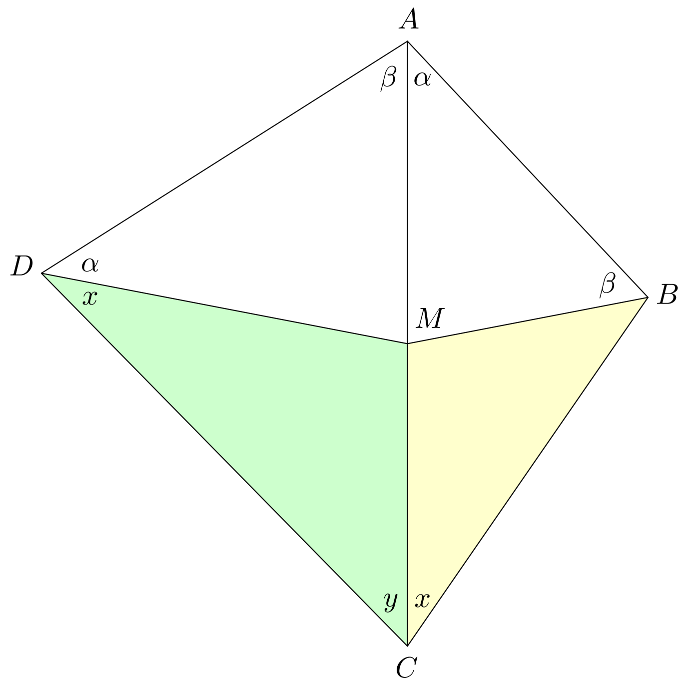
Note that the second line of the above calculation follows from the first because all of $k^2, k^2 + 1, \ldots, k^2 + 2k$ are greater than or equal to k^2 but strictly less than $(k+1)^2$. This completes the induction, thus establishing the claim.

Since f is strictly increasing, the claim implies that $f(n)$ is an integer if and only if $n = k^2 - 1$ for some integer $k \geq 2$.

Recall n is a positive integer less than one million. Hence $f(n)$ is an integer precisely when $n = 2^2 - 1, 3^2 - 1, \ldots, 1000^2 - 1$. Thus there are exactly 999 values of n for which S is an integer. $\square$

6. **Solution 1**

By the alternate segment theorem, since circle AMD is tangent to AB, we may let $\angle MAB = \angle MDA = \alpha$. Analogously, we may let $\angle DAM = \angle ABM = \beta$.



Therefore, $\triangle AMD \sim \triangle BMA$ (AA). Hence

$$\frac{DM}{MA} = \frac{AM}{MB}.$$

Since $AM = MC$, it follows from the above equality that

$$\frac{DM}{MC} = \frac{CM}{MB}. \tag{1}$$

The external angle sums in triangles AMD and AMB yield

$$\angle DMC = \alpha + \beta = \angle CMB.$$

Combining this with (1) implies $\triangle DMC \sim \triangle CMB$ (PAP). Hence we may let $\angle CDM = \angle BCM = x$. Also let $\angle MCD = y$. These allows us to directly compute

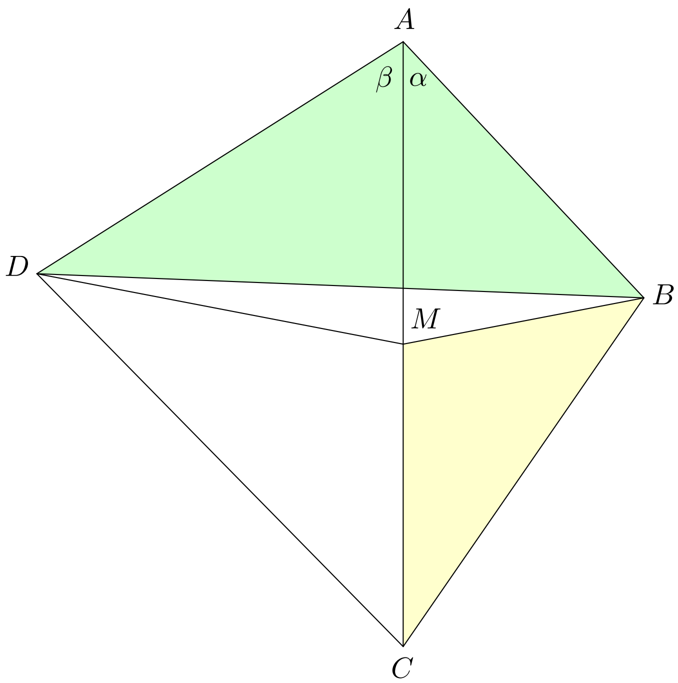
$$\angle DAB + \angle BCD = \alpha + \beta + x + y = 180°,$$

where the last equality is due to the angle sum in $\triangle DMC$.

Since $\angle DAB + \angle BCD = 180°$, we conclude that $ABCD$ is cyclic. $\square$

Solution 2

As in solution (1), we may let $\angle MAB = \angle MDA = \alpha$ and $\angle DAM = \angle ABM = \beta$.



Therefore, $\triangle AMD \sim \triangle BMA$ (AA). Hence

$$\frac{DA}{AM} = \frac{AB}{BM}.$$

Since $AM = MC$, it follows from the above equality that

$$\frac{DA}{CM} = \frac{AB}{MB}. \tag{2}$$

The exterior angle sum in $\triangle AMB$ yields $\angle CMB = \alpha + \beta = \angle DAB$. Combining this with (2) implies $\triangle DAB \sim \triangle CMB$ (PAP). It follows that

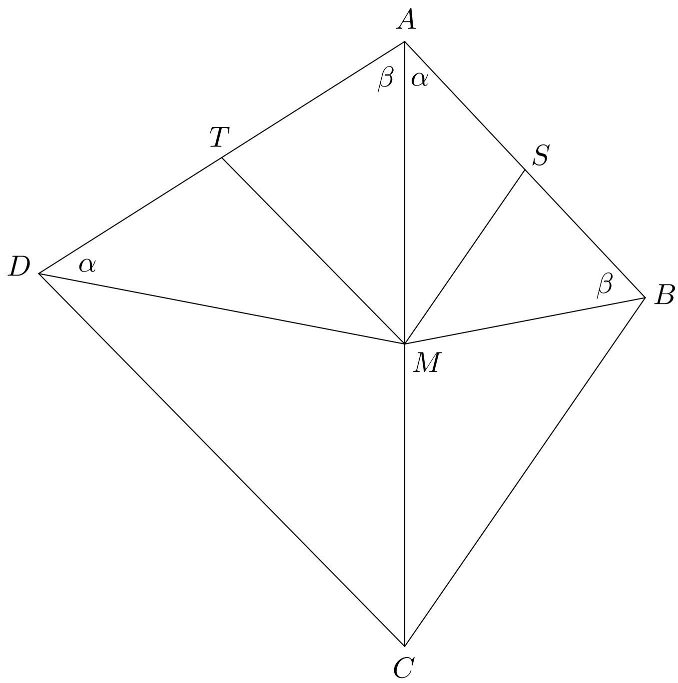
$$\angle BDA = \angle BCM = \angle BCA.$$

Since $\angle BDA = \angle BCA$, it follows that $ABCD$ is cyclic. $\square$

Solution 3

As in solution (1), we have $\triangle AMD \sim \triangle BMA$.

Let S and T be the midpoints of AB and AD, respectively.



Since S and T are corresponding points in similar triangles BMA and AMD, it follows that $\angle MSB = \angle MTA$. This implies that quadrilateral $ASMT$ is cyclic.

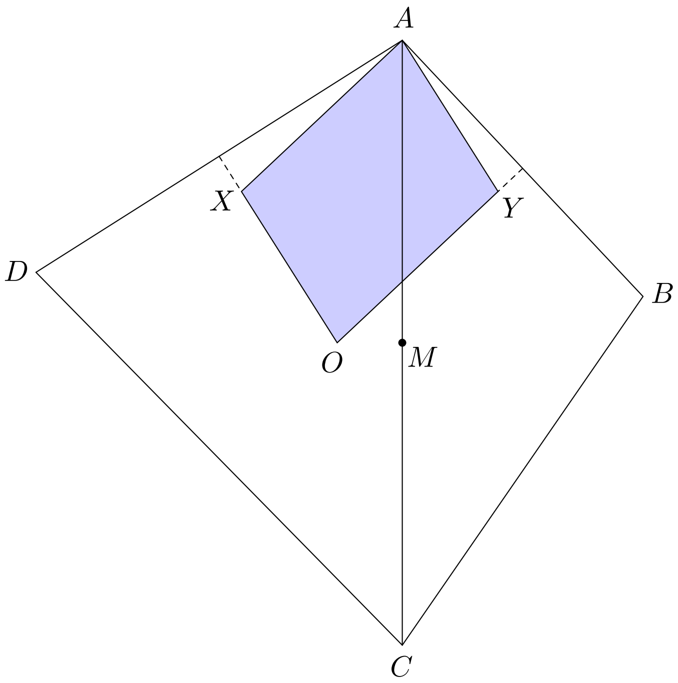
However, since quadrilateral $ABCD$ is the image of $ASMT$ under a dilation of factor 2 about A, it follows that $ABCD$ is also cyclic. $\square$

Comment Here is an alternative way of explaining the above solution. Since $\triangle MDA \sim \triangle MAB$, there is a spiral symmetry centred at M that sends DA to AB.[5] Since T and S are midpoints of DA and AB, respectively, the same spiral symmetry sends T to S, and so sends $\triangle MTA$ to $\triangle MSB$. Thus $\angle MTA = \angle MSB$ which implies that $ASMT$ and hence also $ABCD$ is cyclic.

[5] See the section entitled *Similar Switch* in chapter 5 of *Problem Solving Tactics* published by the AMT.

Solution 4

Let X and R_X denote the centre and radius, respectively, of K_1, and let Y and R_Y denote the centre and radius, respectively, of K_2. Let O be the intersection of the perpendicular bisectors of AD and AB. We shall prove that O is the circumcentre of quadrilateral $ABCD$.



Since AX is a radius of K_1 and AB is a tangent of K_1, we know that $AX \perp AB$. However we also have $YO \perp AB$. Hence $AX \parallel YO$. Similarly $AY \parallel XO$. Hence $AXOY$ is a parallelogram. Thus $YO = AX = R_X$ and $XO = AY = R_Y$.

Consider the reflection in the perpendicular bisector of XY. Let O' be the image of O under this reflection. We claim that $O' = M$. To see this, observe that the segment $O'X$ is the image of OY under the reflection. Hence $O'X = OY = R_X$. Hence O' lies on K_1. Similarly, O' lies on K_2. Hence O' is one of the intersection points of K_1 and K_2. Since O' lies on the same side of XY as O, we have $O' \neq A$. Thus $O' = M$. Furthermore, since $XY \perp AM$, we also have $OM \perp AM$.

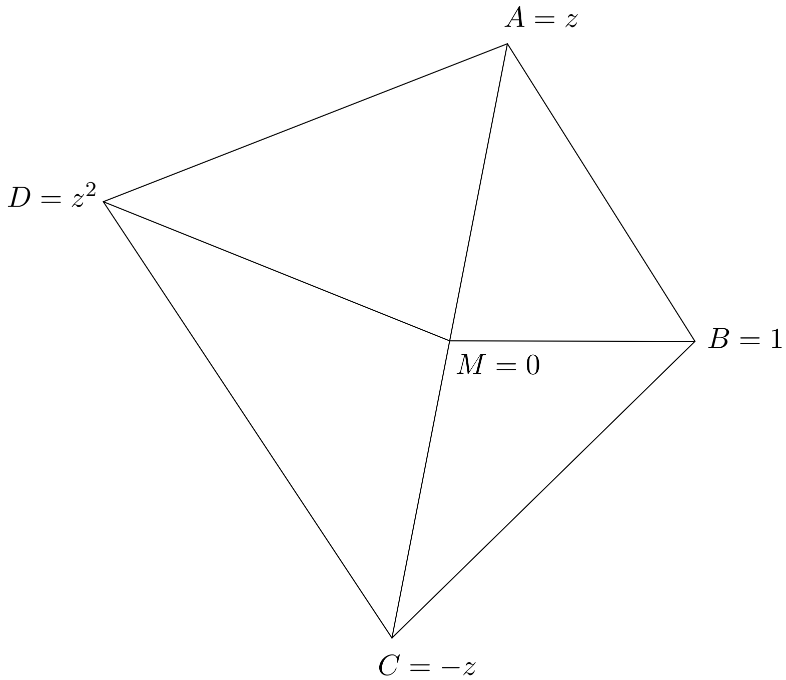
Recall that M is the midpoint of AC. Thus OM is the perpendicular bisector of AC. But OX is the perpendicular bisector of AD. Hence O is the circumcentre of $\triangle ADC$. Similarly O is the circumcentre of $\triangle ABC$. These two deductions imply that O is the circumcentre of quadrilateral $ABCD$, as claimed. $\square$

Solution 5 (Angelo Di Pasquale, Director of Training, AMOC)

This is a computational solution via complex numbers.

Without loss of generality we may assume that points M, B, and A are situated at the complex numbers 0, 1, and z, respectively.

As in solution 1, we have $\triangle BMA \sim \triangle AMD$. Hence D is situated at z^2. Also since M is the midpoint of AC, the point C is situated at $-z$.



Quadrilateral $ABCD$ is cyclic if and only if $\angle DAB + \angle BCD = 180°$. We compute

$$\angle DAB = \arg\left(\frac{B-A}{D-A}\right) = \arg\left(\frac{1-z}{z^2-z}\right) = \arg\left(-\frac{1}{z}\right) = \arg(-1) - \arg z,$$

and

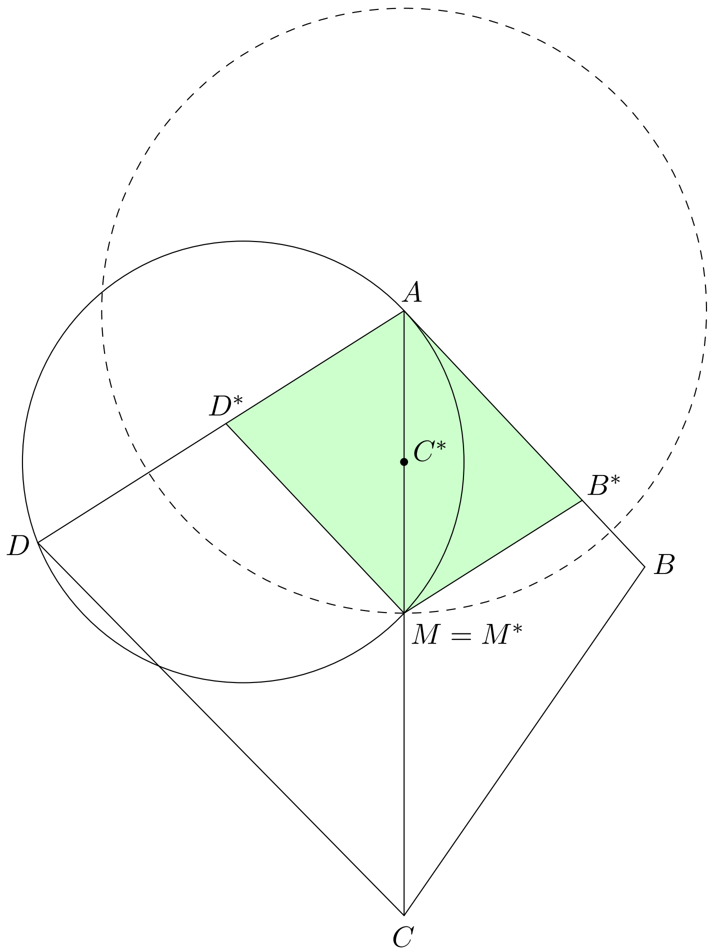
$$\angle BCD = \arg\left(\frac{D-C}{B-C}\right) = \arg\left(\frac{z^2+z}{1+z}\right) = \arg(z).$$

Therefore $\angle DAB + \angle BCD = \arg(-1) = 180°$, as desired. $\square$

Solution 6

This is a solution via inversion.

Consider the inversion about A of radius AM. For any point Z, let Z^* denote its image under the inversion. Clearly $M^* = M$, and C^* is the midpoint of AM.



The lines AD, AM, and AB all pass through A, so they remain fixed under the inversion. Circle ADM is tangent to the line AB. Thus circle ADM is mapped to the line through M that is parallel to AB. From this we deduce that D^* is the intersection of AD and the line through M parallel to AB. Similarly B^* is the intersection of AB and the line through M parallel to AD. Hence AB^*MD^* is a parallelogram.

Any parallogram's diagonals bisect each other. Since C^* is the midpoint of AM, it is also the midpoint of B^*D^*. In particular, B^*, C^*, and D^* are collinear. This implies that circle BCD passes through A. Therefore quadrilateral $ABCD$ is cyclic, as required. $\square$

7. **Solution**

Answers: (a) 32 (b) 0

Let the athletes be $A_1, A_2, \ldots, A_{1000}$ in that order, clockwise around the track.

(a) Starting from A_1 and proceeding clockwise around the track in intervals of 335 metres we meet athletes in the order:

$$A_1, A_{336}, A_{671}, A_6, A_{341}, A_{676}, A_{11}, \ldots, A_{331}, A_{666}, (A_1).$$

Note that since 335 and 1000 are both divisible by 5, only athletes A_i with $i \equiv 1 \pmod 5$ can occur in the above list. Furthermore, the directed distance between every third athlete in the above list is five metres. Thus the above list contains precisely all A_i with $i \equiv 1 \pmod 5$. Hence the list contains exactly 200 different athletes.

Athlete A_1 can either be paired with A_{336} or A_{666}. But once this pairing is chosen, all of the rest of the pairings are forced. Thus there are exactly two ways of pairing up all the athletes A_i for $i \equiv 1 \pmod 5$.

There is nothing particularly special about starting a list with A_1. We could have started four other lists with A_2, A_3, A_4, and A_5, respectively. Analogous arguments show that there are exactly two ways of pairing up the athletes in each such list. Since the five lists are independent, the total number of pairings is $2^5 = 32$. $\square$

(b) Starting from A_1 and proceeding clockwise around the track in intervals of 336 metres we meet athletes in the order:

$$A_1, A_{337}, A_{673}, A_9, A_{345}, A_{681}, A_{17}, \ldots, A_{329}, A_{665}, (A_1).$$

Note that since 336 and 1000 are both divisible by 8, only athletes A_i with $i \equiv 1 \pmod 8$ can occur in the above list. Furthermore, the directed distance between every third athlete in the above list is eight metres. Thus the above list contains precisely all A_i with $i \equiv 1 \pmod 8$. Hence the list contains exactly 125 different athletes. However, since 125 is odd, it is not possible to pair everyone up from the above list. Hence no such pairing is possible. $\square$

Comment (Ivan Guo, AMOC Senior Problems Committee)

For the general problem of n (where n is an even positive integer) athletes standing equally spaced around a circular track of length n metres, it can be shown that the number of ways of dividing the athletes into $\frac{n}{2}$ pairs such that the members of each pair are k metres apart is

$$\begin{cases} 2^{\gcd(k,n)}, & \text{if } \frac{n}{\gcd(k,n)} \text{ is even,} \\ 0, & \text{if } \frac{n}{\gcd(k,n)} \text{ is odd.} \end{cases}$$

8. This was the most difficult problem of the 2017 AMO. Just six contestants managed to solve it completely.

Solution 1

Answer: $n = 2017$

We observe that the function $f(x) = x^2 - 45x + 2$ is a parabola that is symmetric about $x = 22\frac{1}{2}$. Hence for any real number x we have

$$f(x) = f(45 - x). \tag{1}$$

Suppose that $n \mid f(k)$ where k is an integer satisfying $1 \leq k \leq n$. Using (1), it follows that $n \mid f(45 - k)$. However, since f is a polynomial, we know that

$$a \equiv b \pmod{n} \quad \Rightarrow \quad f(a) \equiv f(b) \pmod{n}. \tag{2}$$

Hence if $j \equiv 45 - k \pmod{n}$, where $1 \leq j \leq n$, then from (2) we have

$$f(j) \equiv f(45 - k) \equiv 0 \pmod{n}.$$

Since exactly one of the numbers $f(1), f(2), \ldots, f(n)$ is divisible by n, we have

$$\begin{aligned} k &\equiv 45 - k \pmod{n} \\ \Rightarrow \quad 2k &\equiv 45 \pmod{n}. \end{aligned} \tag{3}$$

We also know that $n \mid f(k)$, and so we may compute as follows.

$$\begin{aligned} & k^2 - 45k + 2 \equiv 0 \pmod{n} \\ \Rightarrow \quad & (2k)^2 - 90 \times 2k + 8 \equiv 0 \pmod{n} \\ \Rightarrow \quad & (45)^2 - 90 \times 45 + 8 \equiv 0 \pmod{n} \quad \text{(from (3))} \\ \Leftrightarrow \quad & -2017 \equiv 0 \pmod{n} \end{aligned}$$

Hence $n \mid 2017$. Since $n \geq 2$ and 2017 is prime, we have $n = 2017$. However, we still must check whether or not $n = 2017$ actually works.

Suppose that $2017 \mid f(k)$, where $1 \leq k \leq 2017$. We compute as follows.

$$\begin{aligned} & k^2 - 45k + 2 \equiv 0 \pmod{2017} \\ \Leftrightarrow \quad & 4k^2 - 180k + 8 \equiv 0 \pmod{2017} \quad \text{(since 2017 is odd)} \\ \Leftrightarrow \quad & (2k - 45)^2 \equiv 0 \pmod{2017} \\ \Leftrightarrow \quad & (2k - 45) \equiv 0 \pmod{2017} \quad \text{(since 2017 is prime)} \\ \Leftrightarrow \quad & 2k \equiv 45 \pmod{2017} \\ & \equiv 2062 \pmod{2017} \\ \Leftrightarrow \quad & k \equiv 1031 \pmod{2017} \quad \text{(since 2017 is odd)} \\ \Leftrightarrow \quad & k = 1031 \quad \text{(since } 1 \leq k \leq 2017\text{)} \end{aligned}$$

This shows that $k = 1031$ is the only integer with $1 \leq k \leq 2017$ such that $f(k)$ is divisible by 2017. ☐

Solution 2 (Angelo Di Pasquale, Director of Training, AMOC)

Note that if $x \equiv y \pmod{n}$, then it follows that $f(x) \equiv f(y) \pmod{n}$. Therefore, we are seeking all n such that $f(x) \equiv 0 \pmod{n}$ has a unique solution modulo n.

Suppose that $f(k) = an$ for some integer a. Using the quadratic formula, we find that

$$k = \frac{45 \pm \sqrt{2017 + 4an}}{2}. \tag{1}$$

Hence, $2017 + 4an$ is an odd perfect square. So if one root of the quadratic is an integer, then so is the other. By the condition of the problem, this implies that

$$\frac{45 + \sqrt{2017 + 4an}}{2} \equiv \frac{45 - \sqrt{2017 + 4an}}{2} \pmod{n}.$$

Transferring everything in the above congruence to the left yields

$$\sqrt{2017 + 4an} \equiv 0 \pmod{n}.$$

Squaring the above yields $2017 \equiv 0 \pmod{n}$. Since 2017 is prime and $n \geq 2$, it follows that $n = 2017$.

Conversely, if $n = 2017$, then the quadratic formula (1) tells us that for k to be an integer, we require $1 + 4a = 2017j^2$ for some odd integer $j = 2i + 1$. Substituting this into the equation yields $k = 1031 + 2017i$ or $k = -986 - 2017i$. So the only such value of k in the required range is $k = 1031$, which corresponds to $i = 0$, $j = 1$ and $a = 504$. $\square$

AUSTRALIAN MATHEMATICAL OLYMPIAD STATISTICS

Score Distribution/Problem

Number of Students/Score	Problem Number							
	1	2	3	4	5	6	7	8
0	0	4	34	81	4	33	21	82
1	6	9	16	5	0	14	2	10
2	7	5	12	0	1	8	1	1
3	0	1	7	0	3	1	1	0
4	1	5	0	0	4	1	5	3
5	3	7	2	3	0	1	22	2
6	7	26	7	2	37	1	0	0
7	80	47	26	13	55	45	52	6
Average	6.2	5.4	2.8	1.2	6.1	3.5	4.8	0.7