AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

A DEPARTMENT OF THE AUSTRALIAN MATHEMATICS TRUST

DAY 1
Tuesday, 6 February 2018
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Find all pairs of positive integers (n, k) such that

$$n! + 8 = 2^k.$$

(If n is a positive integer, then $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.)

2. Consider a line with $\frac{1}{2}(3^{100} + 1)$ equally spaced points marked on it.

Prove that 2^{100} of these marked points can be coloured red so that no red point is at the same distance from two other red points.

3. Let $ABCDEFGHIJKLMN$ be a regular tetradecagon.

Prove that the three lines AE, BG and CK intersect at a point.

(A *regular tetradecagon* is a convex polygon with 14 sides, such that all sides have the same length and all angles are equal.)

4. Find all functions f defined for real numbers and taking real numbers as values such that

$$f(xy + f(y)) = yf(x)$$

for all real numbers x and y.

DAY 2

Wednesday, 7 February 2018
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 1$ and, for $n \geq 2$,

$$a_n = (a_1 + a_2 + \cdots + a_{n-1}) \times n.$$

Prove that a_{2018} is divisible by 2018^2.

6. Let P, Q and R be three points on a circle $\mathcal{C}$, such that $PQ = PR$ and $PQ > QR$. Let $\mathcal{D}$ be the circle with centre P that passes through Q and R. Suppose that the circle with centre Q and passing through R intersects $\mathcal{C}$ again at X and $\mathcal{D}$ again at Y.

Prove that P, X and Y lie on a line.

7. Let $b_1, b_2, b_3, \ldots$ be a sequence of positive integers such that, for each positive integer n, b_{n+1} is the square of the number of positive factors of b_n (including 1 and b_n). For example, if $b_1 = 27$, then $b_2 = 4^2 = 16$, since 27 has four positive factors: 1, 3, 9 and 27.

Prove that if $b_1 > 1$, then the sequence contains a term that is equal to 9.

8. Amy has a number of rocks such that the mass of each rock, in kilograms, is a positive integer. The sum of the masses of the rocks is 2018 kilograms. Amy realises that it is impossible to divide the rocks into two piles of 1009 kilograms.

What is the maximum possible number of rocks that Amy could have?