AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

A DEPARTMENT OF THE AUSTRALIAN MATHEMATICS TRUST

DAY 1

Tuesday, 6 February 2018

Time allowed: 4 hours

No calculators are to be used.

Each question is worth seven points.

1. Find all pairs of positive integers (n, k) such that

$$n! + 8 = 2^k.$$

(If n is a positive integer, then $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.)

2. Consider a line with $\frac{1}{2}(3^{100} + 1)$ equally spaced points marked on it.

Prove that 2^{100} of these marked points can be coloured red so that no red point is at the same distance from two other red points.

3. Let $ABCDEFGHIJKLMN$ be a regular tetradecagon.

Prove that the three lines AE, BG and CK intersect at a point.

(A *regular tetradecagon* is a convex polygon with 14 sides, such that all sides have the same length and all angles are equal.)

4. Find all functions f defined for real numbers and taking real numbers as values such that

$$f(xy + f(y)) = yf(x)$$

for all real numbers x and y.

DAY 2
Wednesday, 7 February 2018
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 1$ and, for $n \geq 2$,

$$a_n = (a_1 + a_2 + \cdots + a_{n-1}) \times n.$$

Prove that a_{2018} is divisible by 2018^2.

6. Let P, Q and R be three points on a circle $\mathcal{C}$, such that $PQ = PR$ and $PQ > QR$. Let $\mathcal{D}$ be the circle with centre P that passes through Q and R. Suppose that the circle with centre Q and passing through R intersects $\mathcal{C}$ again at X and $\mathcal{D}$ again at Y.

Prove that P, X and Y lie on a line.

7. Let $b_1, b_2, b_3, \ldots$ be a sequence of positive integers such that, for each positive integer n, b_{n+1} is the square of the number of positive factors of b_n (including 1 and b_n). For example, if $b_1 = 27$, then $b_2 = 4^2 = 16$, since 27 has four positive factors: 1, 3, 9 and 27.

Prove that if $b_1 > 1$, then the sequence contains a term that is equal to 9.

8. Amy has a number of rocks such that the mass of each rock, in kilograms, is a positive integer. The sum of the masses of the rocks is 2018 kilograms. Amy realises that it is impossible to divide the rocks into two piles of 1009 kilograms.

What is the maximum possible number of rocks that Amy could have?

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The Mathematics/Informatics Olympiads are supported by the Australian Government through the National Innovation and Science Agenda.

1. **Solution 1** (Evgeniya Artemova, year 11, Presbyterian Ladies' College, VIC)

   Answers $(n, k) = (4, 5)$ and $(5, 7)$.

   For reference the given equation is

$$n! + 8 = 2^k.$$

Case 1 $n \geq 6$

Observe that $n!$ is a multiple of $6! = 2^4 \times 3^2 \times 5$. Hence $n! = 16x$ for some positive integer x. Therefore

$$n! + 8 = 8(2x + 1).$$

But the RHS of the above equation cannot be a power of 2 because $2x+1$ is an odd integer that is greater than 1. Hence there are no solutions in this case.

Case 2 $n \leq 5$

We simply tabulate the values of $n! + 8$ and check which ones are powers of 2.

n	$n! + 8$	power of 2?
1	9	no
2	10	no
3	14	no
4	32	2^5
5	128	2^7

This yields the solutions given at the outset. ☐

Solution 2 (Andres Buritica, year 9, Scotch College, VIC)

For reference the given equation is

$$n! + 8 = 2^k.$$

Case 1 $n \geq 6$

We have $2^4 \mid n!$ and $2^3 \parallel 8$. Thus $2^3 \parallel n! + 8$.[1] Hence $2^3 \parallel 2^k$, and so $k = 3$. But this implies $n! = 0$, which is impossible. So there are no solutions in this case.

Case 2 $n \leq 3$

We have $2^3 \nmid n!$ and $2^3 \mid 8$. Hence $2^3 \nmid 2^k$, and so $k < 3$. It follows that $n! < 0$, which is impossible. So there are no solutions in this case.

Case 3 $n = 4$ or 5

If $n = 4$ then $k = 5$, and if $n = 5$ then $k = 7$. $\square$

[1] For a prime number p and integers $k \geq 0$ and $N \geq 1$, the notation $p^k \parallel N$ means that $p^k \mid N$ but $p^{k+1} \nmid N$. Put another way, it means that the exponent of p in the prime factorisation of N is k.

2. **Solution 1** (William Hu, year 12, Christ Church Grammar School, WA)

Given some equally spaced points marked on a line, if we colour some of them red, we say the colouring is *good* if no red point is equidistant from two other red points. The result is the special case $n = 100$ of the following more general claim.

Claim For each positive integer n, if a line has $\frac{1}{2}(3^n + 1)$ equally spaced points marked on it, then there is a good colouring of 2^n of those points.

The proof is by induction on n.

The base case $n = 1$ holds as we simply colour each of the $\frac{1}{2}(3^1 + 1) = 2^1$ points red.

For the inductive step, suppose that the claim is true for some positive integer n. Let $k = \frac{1}{2}(3^n + 1)$. Note that $3k - 1 = \frac{1}{2}(3^{n+1} + 1)$. To prove the claim for $n + 1$ we divide the $3k - 1$ points into three groups from left to right as follows.

$$\underbrace{k \text{ points}}_{\text{group A}} \qquad \underbrace{k-1 \text{ points}}_{\text{group B}} \qquad \underbrace{k \text{ points}}_{\text{group C}}$$

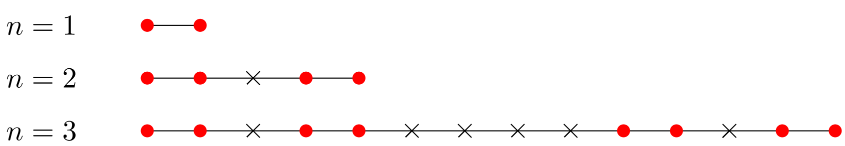
Using the inductive assumption we colour 2^n points in group A red so that no red point in A is equidistant from two other red points in A. Similarly we colour 2^n points in C red so that no red point in C is equidistant from two other red points in C. All points in B are left uncoloured.

A total of $2^n + 2^n = 2^{n+1}$ points have been coloured red in the union of the three groups. Suppose, for the sake of contradiction, that this colouring is not good. Then there are three red points W, X, Y in that order from left to right, where $WX = XY$. Without loss of generality X is in A. Thus W is also in A. However,

- Y is not in A, from the inductive assumption,
- Y is not in B, because no point in Y is coloured red, and
- Y is not in C, because $WX \leq k - 1$ and $XY \geq k$.

This contradiction completes the induction and the proof. $\square$

Comment In order to illustrate the above proof, here are some iterations of the inductive construction.



Solution 2 (Sharvil Kesarwani, year 11, Merewether High School, NSW)

Without loss of generality, we identify the $\frac{1}{2}(3^{100}+1)$ equally spaced points with the integers from 0 up to $\frac{1}{2}(3^{100}-1)$ on the real number line.

Colour red each integer of the form

$$N=\sum_{i=0}^{99} d_i \cdot 3^i$$

where $d_i \in \{0,1\}$ for $0 \le i \le 99$. In this way exactly 2^{100} integers are coloured red. Note that these are precisely the integers in the range from 0 to $\frac{1}{2}(3^{100}-1)$ which do not contain the digit 2 in their ternary (base-3) representations.

Suppose, for the sake of contradiction, that one red integer B is equidistant from two other red integers A and C, where $A<B<C$. Thus $C-B=B-A$, which is the same as $2B=A+C$.

Let

$$A=\sum_{i=0}^{99} a_i \cdot 3^i, \quad B=\sum_{i=0}^{99} b_i \cdot 3^i, \quad \text{and} \quad C=\sum_{i=0}^{99} c_i \cdot 3^i$$

be the ternary representations of A, B, and C, where $a_i, b_i, c_i \in \{0,1\}$ for $0 \le i \le 99$. From $2B=A+C$ we have

$$\sum_{i=0}^{99} 2b_i \cdot 3^i=\sum_{i=0}^{99}(a_i+c_i)3^i.$$

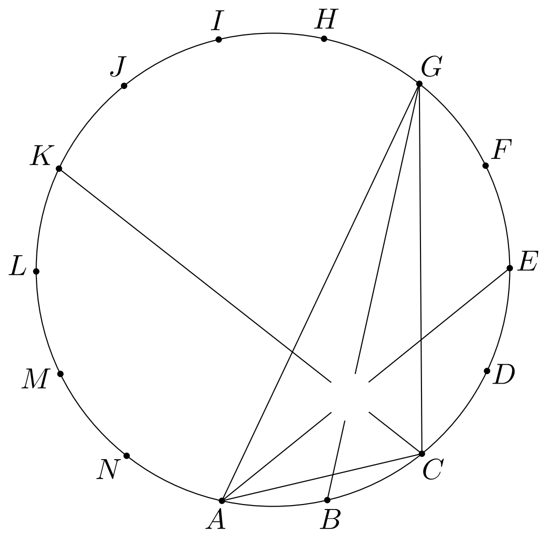
Since $a_i, b_i, c_i \in \{0,1\}$, we have $2b_i \in \{0,2\}$ and $a_i+c_i \in \{0,1,2\}$. Therefore both sides of the above equation are the ternary representation of the same number. It follows that $2b_i=a_i+c_i$ for each i.

If $b_i=0$, then $a_i=c_i=0$. And if $b_i=1$, then $a_i=c_i=1$. Either way, A and C have the same ternary digits, and so $A=C$. This contradicts $A<C$, and completes the proof. $\square$

3. **Solution 1** (Elizabeth Yevdokimov, year 9, St Ursula's College, QLD)

Since the tetradecagon is regular, it has a circumcircle.

Form triangle AGC by joining the segments AC, CG, and GA.



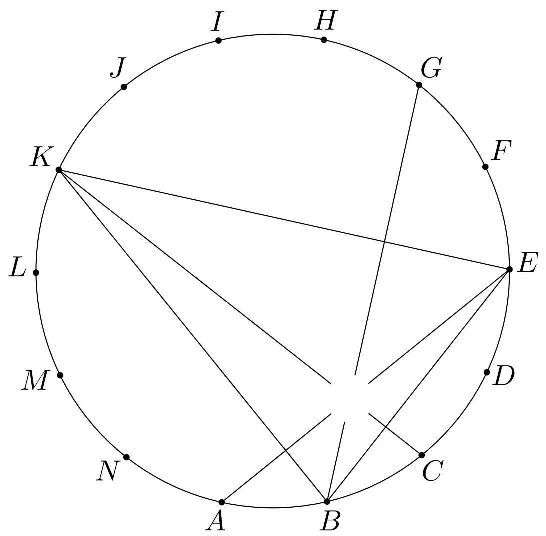
Since the vertices of the tetradecagon are equally spaced around the circle, and equal length arcs subtend equal angles, we have

$$\angle CAE = \angle EAG, \quad \angle AGB = \angle BGC, \quad \text{and} \quad \angle GCK = \angle KCA.$$

Hence AE, BG, and CK are the internal bisectors of the angles of $\triangle AGC$. As such they are concurrent at the incentre of $\triangle AGC$. $\square$

Solution 2 (Based on the solution by Ethan Ryoo, year 9, Knox Grammar School, NSW)

As in solution 1, we consider $A, B, \ldots, N$ as being 14 equally spaced points around a circle. Form triangle BEK by joining segments BE, EK, and KB.



By symmetry we have parallel chords $BG \parallel CF \parallel DE$. Since DK is a diameter of the circle, we have $DE \perp EK$. Hence $BG \perp EK$.

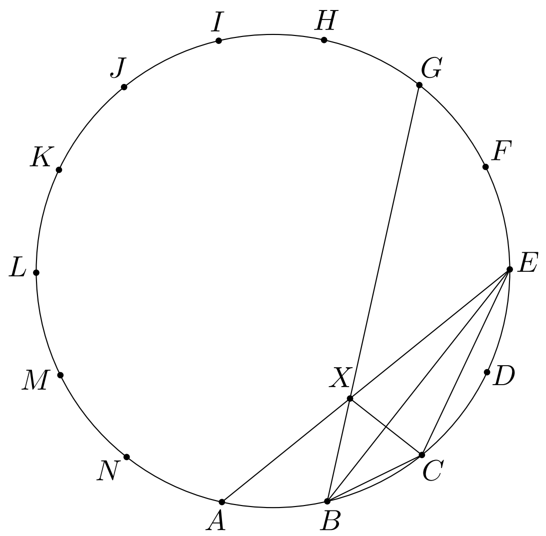
Similarly we have $EA \parallel DB$ and $DB \perp BK$. Hence $EA \perp BK$.

Finally we have $KC \parallel LB$ and $LB \perp BE$ (LE is a diameter). Hence $KC \perp BE$.

We have shown that BG, EA, and KC are altitudes of $\triangle BEK$. As such they are concurrent at the orthocentre of $\triangle BEK$. $\square$

Solution 3 (Ethan Tan, year 12, Cranbrook School, NSW)

As in solution 1, we consider $A, B, \ldots, N$ as being 14 equally spaced points around a circle. Let X be the intersection of lines AE and BG. It suffices to show that C, X, and K are collinear.



Since the vertices of the tetradecagon are equally spaced around the circle, and equal length arcs subtend equal angles, we have

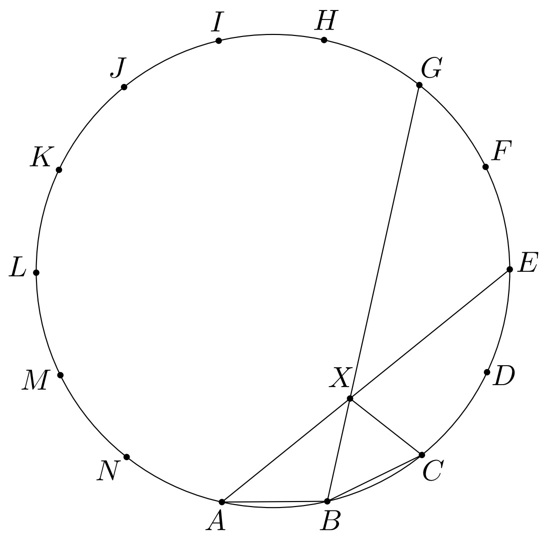
$$\angle XEB = \angle AEB = \angle BEC \quad \text{and} \quad \angle CBE = \angle EBG = \angle EBX.$$

It follows that $\triangle BEX \equiv \triangle BEC$ (ASA). Hence $BCEX$ is a kite with $BE \perp CX$.

We also have parallel chords $BE \parallel CD$. Hence $CD \perp CX$. But $CD \perp CK$ because DK is a diameter of the circle. Thus $CK \parallel CX$, from which it follows that C, X, and K are collinear, as desired. $\square$

Solution 4 (Yifan Guo, year 12, Glen Waverley Secondary College, VIC)

As in solution 1, we consider $A, B, \ldots, N$ as being 14 equally spaced points around a circle. Let X be the intersection of lines AE and BG. It suffices to show that C, X, and K are collinear. Let α satisfy $14\alpha = 180°$.



Each of the 14 equal sides of the tetradecagon subtends an angle of $360°/14$ with the centre of the circle. Hence each of these sides subtends an angle of $\alpha = 180°/14$ with a point on the major arc opposite the side. Using this we have

$$\angle BAE = \angle BAC + \angle CAD + \angle DAE = 3\alpha.$$

Similar calculations yield

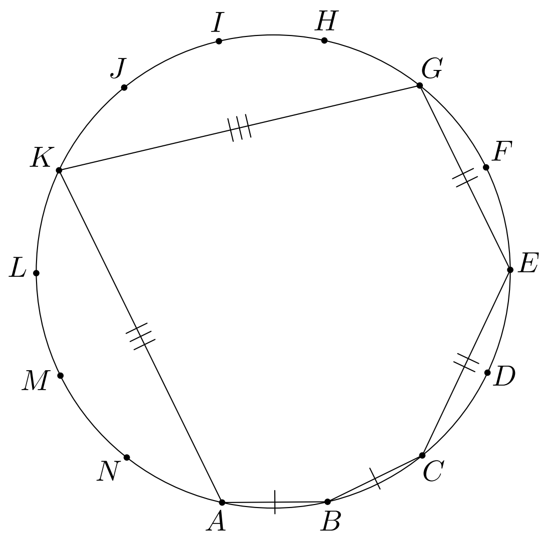
$$\angle GBA = 8\alpha \quad \text{and} \quad \angle KCB = 5\alpha.$$

A consideration of the angle sum in $\triangle ABX$ yields $\angle BXA = 3\alpha$. Hence $BX = BA$. However $BA = BC$. Hence $\triangle BXC$ is isosceles with apex B. Since $\angle CBX = 4\alpha$, it follows that $\angle BXC = \angle XCB = 5\alpha$.

Finally, since $\angle XCB = 5\alpha = \angle KCB$, it follows that C, X, and K are collinear, as desired. $\square$

Solution 5 (Mikhail Savkin, year 10, Gosford High School, NSW)

As in solution 1, we consider $A, B, \ldots, N$ as being 14 equally spaced points around a circle. Form the hexagon $ABCEGK$.



We quote a theorem that will help us solve the problem.

Theorem If $UVWXYZ$ is a convex cyclic hexagon, then it's main diagonals UX, VY, and WZ are concurrent if and only if

$$UV \cdot WX \cdot YZ = VW \cdot XY \cdot ZA.$$

This result is found in configuration C6 in chapter 5 of *Problem Solving Tactics*.[2]

We apply the theorem as follows. From the equal spacing of points of the tetradecagon around the circle, we deduce that

$$AB = BC, \quad CE = EG, \quad \text{and} \quad GK = KA.$$

Since $ABCEGK$ is a convex cyclic hexagon with the property that

$$AB \cdot CE \cdot GK = BC \cdot EG \cdot KA,$$

the result follows. $\square$

[2] Published by the AMT

4. **Solution 1** (William Hu, year 12, Christ Church Grammar School, WA)

Answers $f(x) = 0$ and $f(x) = 1 - x$

For reference the given functional equation is

$$f(xy + f(y)) = yf(x) \tag{1}$$

for all real numbers x and y.

Put $y = 0$ in (1) to find

$$f(f(0)) = 0. \tag{2}$$

Next put $x = 0$ and $y = f(0)$ in (1), and use (2) to deduce

$$f(0) = f(0)^2.$$

Thus $f(0) = 0$ or 1.

Case 1 $f(0) = 0$

Put $x = 0$ in (1) to find

$$f(f(y)) = 0 \tag{3}$$

for all $y \in \mathbb{R}$.

Suppose there exists a real number c with $f(c) \neq 0$. Then if we put $x = c$ in (1), we see that RHS(1) covers all real numbers, and so f is surjective. But if f is surjective, then so is $f \circ f$, which contradicts (3). Hence no such c exists. Thus $f(x) = 0$ for all real numbers x. This function obviously satisfies (1).

Case 2 $f(0) = 1$

Putting $x = 0$ into (1) yields

$$f(f(y)) = y \tag{5}$$

for all real numbers y. It follows that $f(1) = f(f(0)) = 0$.

Putting $x = 1$ in (1) yields

$$f(y + f(y)) = 0 = f(1) \quad \Rightarrow \quad f(f(y + f(y))) = f(f(1)).$$

With the help of (5) this implies $y + f(y) = 1$, and so $f(y) = 1 - y$.

It only remains to verify that $f(x) = 1 - x$ satisfies (1). Indeed we have

$$\text{LHS}(1) = 1 - (xy + (1 - y)) = y(1 - x) = \text{RHS}(1).$$

Hence $f(x) = 1 - x$ is also a solution to the problem. $\square$

Solution 2 (William Steinberg, year 10, Scotch College WA)

For reference the given functional equation is

$$f(xy + f(y)) = yf(x) \tag{1}$$

for all real numbers x and y.

Put $x = 0$ in (1) to find

$$f(f(y)) = yf(0), \tag{2}$$

for all real numbers y.

Putting $y = 1$ in (1) yields

$$\begin{aligned}
& f(x + f(1)) = f(x) \\
\Rightarrow\quad & f(f(x + f(1)) = f(f(x)) \\
\Rightarrow\quad & (x + f(1))f(0) = xf(0) \qquad \text{(from (2))} \\
\Rightarrow\quad & f(0)f(1) = 0.
\end{aligned} \tag{3}$$

Two cases follow from (3).

Case 1 $f(0) = 0$

Equation (2) now becomes

$$f(f(y)) = 0 \tag{4}$$

for all real numbers y.

Applying f to both sides of (1) and then using (4) yields

$$f(yf(x)) = 0. \tag{5}$$

Suppose that there exists a real number c with $f(c) \neq 0$. Then putting $x = c$ and $y = c/f(c)$ in (5), we deduce that $f(c) = 0$. This is a contradiction. Thus no such c exists. Hence $f(x) = 0$ for all real numbers x.

Case 2 $f(0) \neq 0$

It follows from (3) that $f(1) = 0$.

Setting $x = 1$ in (1), and using $f(1) = 0$ yields

$$\begin{aligned}
& f(y + f(y)) = 0 \\
\Rightarrow\quad & f(f(y + f(y)) = f(0) \\
\Rightarrow\quad & (y + f(y))f(0) = f(0) \quad \text{(from (2))} \\
\Rightarrow\quad & y + f(y) = 1. \qquad \text{(since } f(0) \neq 0\text{)}
\end{aligned}$$

Thus $f(y) = 1 - y$.

As in solution 1, we check that $f(x) = 0$ and $f(x) = 1 - x$ each satisfy the given functional equation. ☐

5. **Solution 1** (Vicky Feng, year 11, Methodist Ladies' College, NSW)

We are given

$$a_n = n(a_1 + a_2 + \cdots + a_{n-1}) \tag{1}$$

for any integer $n \geq 2$.

Replacing n with $n-1$ in (1), we find

$$a_{n-1} = (n-1)(a_1 + a_2 + \cdots + a_{n-2}) \tag{2}$$

for any integer $n \geq 3$.

Substituting (2) into (1) yields

$$\begin{aligned}
a_n &= n(a_1 + a_2 + \cdots + a_{n-2} + a_{n-1}) \\
&= n(a_1 + a_2 + \cdots + a_{n-2} + (n-1)(a_1 + a_2 + \cdots + a_{n-2})) \\
&= n^2(a_1 + a_2 + \cdots + a_{n-2}).
\end{aligned}$$

Hence a_n is divisible by n^2 for each integer $n \geq 3$. In particular, a_{2018} is divisible by 2018^2. $\square$

Comment We did not use the initial condition $a_1 = 1$. Hence the result is true for any integer a_1.

Solution 2 (Xinyue (Alice) Zhang, year 12, A. B. Paterson College, QLD)

We prove an even stronger result, namely a_{2018} is divisible by 2018^3.

First we prove by induction that $a_1 + a_2 + \cdots + a_{n-1} = n!/2$ for all integers $n \geq 2$.

The base case is true because $a_1 = 1 = \frac{2!}{2}$.

For the inductive step, assume that $a_1 + a_2 + \cdots + a_{n-1} = n!/2$ for some integer $n \geq 2$. Then

$$\begin{aligned} a_1 + a_2 + \cdots + a_{n-1} + a_n &= a_1 + a_2 + \cdots + a_{n-1} + n(a_1 + a_2 + \cdots + a_{n-1}) \\ &= \frac{n!}{2} + \frac{n \cdot n!}{2} \\ &= \frac{(n+1) \cdot n!}{2} \\ &= \frac{(n+1)!}{2} \end{aligned}$$

which completes the induction.

Using this result, it follows that

$$\begin{aligned} a_{2018} &= 2018(a_1 + \cdots + a_{2017}) \\ &= 2018(2018 \times 2017 \times \cdots \times 3) \\ &= 2018^2(2017 \times 2016 \times \cdots \times 1009 \times \cdots \times 4 \times 3) \end{aligned}$$

which is obviously a multiple of 2018^2. However $2017 \times 2016 \times \cdots \times 1009 \times \cdots \times 4 \times 3$ is divisible by both 2 and 1009, and so it is also divisible by 2018. Therefore a_{2018} is divisible by 2018^3. $\square$

Solution 3 (Andres Buritica, year 9, Scotch College, VIC)

We are given

$$\frac{a_n}{n} = a_1 + a_2 + \cdots + a_{n-1} \tag{1}$$

for any integer $n \geq 2$.

Replacing n with $n+1$ in (1), we find

$$\begin{aligned} \frac{a_{n+1}}{n+1} &= a_1 + a_2 + \cdots + a_{n-1} + a_n \\ &= \frac{a_n}{n} + a_n \\ \Rightarrow \qquad \frac{a_{n+1}}{a_n} &= \frac{(n+1)^2}{n} \end{aligned} \tag{2}$$

for any integer $n \geq 2$.

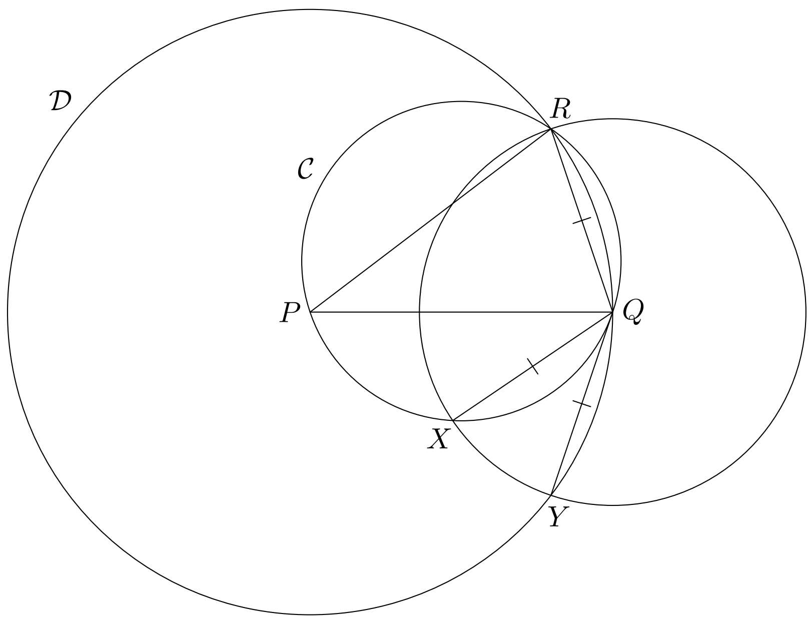
Multiplying all instances of equality (2) together for $n = m-1, m-2, \ldots, 2$, where $m \geq 3$, yields a product that telescopes as follows.

$$\begin{aligned} \frac{a_m}{a_{m-1}} \times \frac{a_{m-1}}{a_{m-2}} \times \cdots \times \frac{a_3}{a_2} &= \frac{m^2}{m-1} \times \frac{(m-1)^2}{m-2} \times \cdots \times \frac{3^2}{2} \\ \Rightarrow \qquad \frac{a_m}{a_2} &= \frac{m^2(m-1)!}{4} \\ \Rightarrow \qquad a_m &= \frac{m^2(m-1)!}{2} \end{aligned}$$

In particular $a_{2018} = (2018)^2 \times \frac{2017!}{2}$, which is a multiple of 2018^2. $\square$

6. **Solution 1** (William Steinberg, year 10, Scotch College, WA)

Since R, X, and Y all lie on a circle centred at Q, we have $QR = QX = QY$.



In circle $\mathcal{D}$, chords QY and QR have equal length. Hence they subtend equal angles at the centre P of this circle. Thus

$$\angle YPQ = \angle QPR. \tag{1}$$

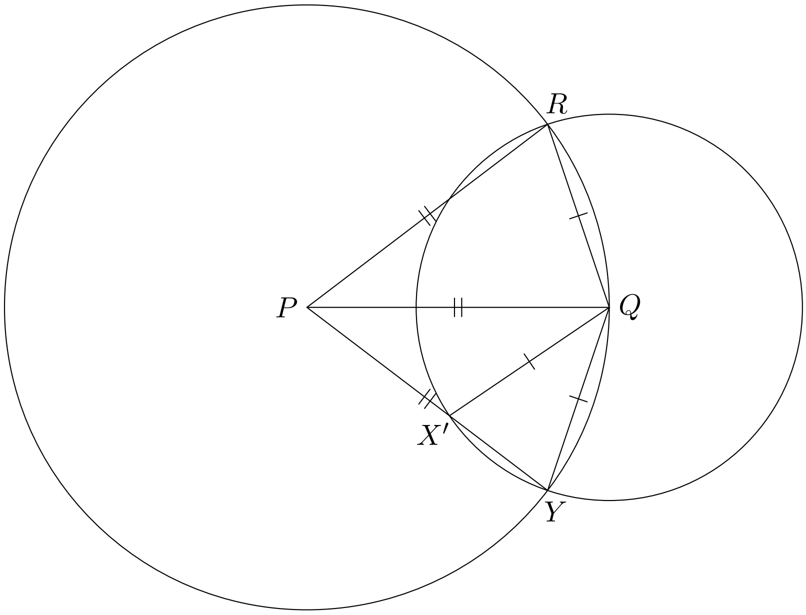
In circle $\mathcal{C}$, chords QX and QR have equal length. Hence they subtend equal angles at the point P on the circumference of this circle. Thus

$$\angle XPQ = \angle QPR. \tag{2}$$

From (1) and (2) we have $\angle XPQ = \angle YPQ$. Thus P, X, and Y are collinear. $\square$

Solution 2 (Matthew Kerr, year 12, St Anthony's Catholic College, QLD)

Let the circle centred at Q and passing through R intersect the line PY for a second time at X'. It suffices to prove that $PRQX'$ is cyclic as this implies $X' = X$.



As P and Q are centres of circles, we have $PR = PQ = PY$ and $QR = QX' = QY$. Hence $\triangle PQR \equiv \triangle PQY$ (SSS). Thus

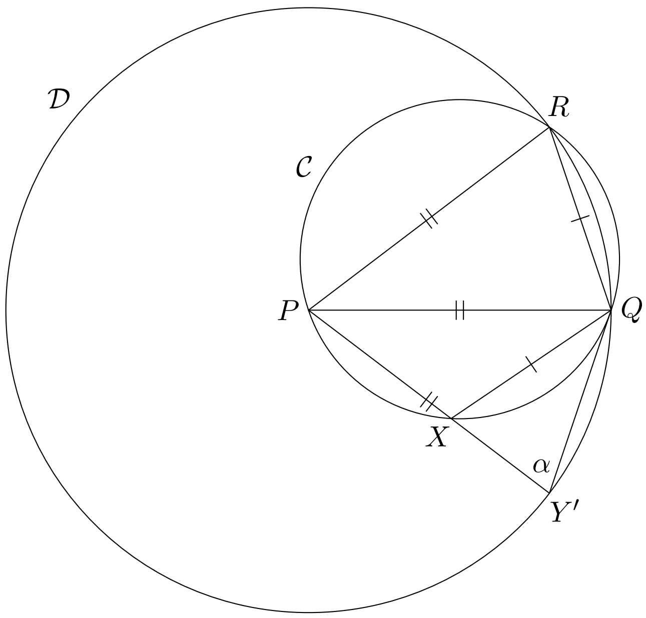
$$\angle PRQ = \angle QYP = \angle QYX' = \angle YX'Q$$

where the last equality is due to $QX' = QY$.

Since $\angle PRQ = \angle YX'Q$, it follows that $PRQX'$ is cyclic, as desired. $\square$

Solution 3 (Sharvil Kesarwani, year 11, Merewether High School, NSW)

Let Y' be the intersection of ray PX with circle $\mathcal{D}$. It suffices to prove that $Y = Y'$. Let $\angle QY'X = \angle QY'P = \alpha$.



Observe that $PY' = PQ = PR$ because they are radii of circle $\mathcal{D}$. It follows that $\angle PQY' = \angle QY'P = \alpha$. Considering the angle sum in $\triangle PQY$ yields

$$\angle Y'PQ = 180^\circ - 2\alpha.$$

Observe that $QX = QR$ because X lies on the circle centred at Q and passing through R. Since QX and QR are equal length chords in circle $\mathcal{C}$, they subtend equal angles at the point P on the circumference of this circle. Hence

$$\angle QPR = \angle XPQ = \angle Y'PQ = 180^\circ - 2\alpha.$$

Since $\triangle PQR$ is isosceles with apex P, the angle sum in this triangle yields

$$\angle PRQ = \angle RQP = \alpha.$$

Since $PRQX$ is a cyclic quadrilateral, we have

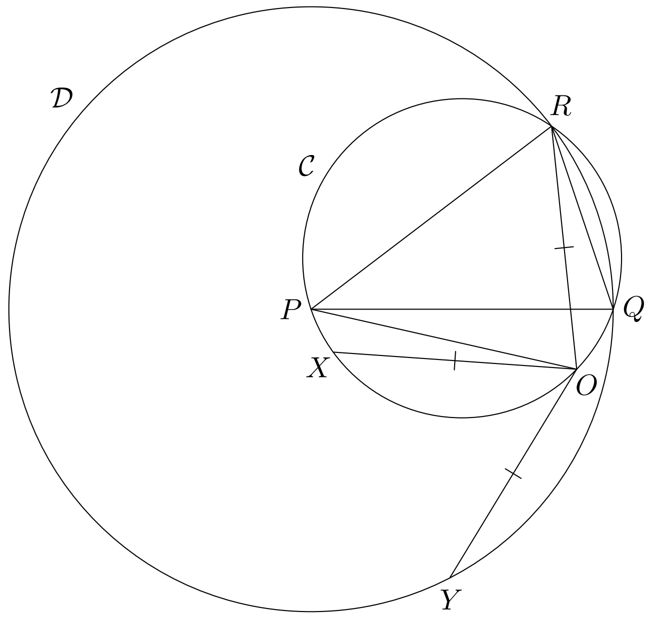
$$\angle Y'XQ = \angle PRQ = \alpha = \angle QY'X.$$

Hence $\triangle QXY'$ is isosceles with apex Q. Thus $QY' = QX = QR$, and so Y' lies on the circle centred at Q and passing through R. Since Y' and R are on opposite sides of the line PQ, it follows that $Y = Y'$. $\square$

Solution 4 (Alan Offer, AMOC Senior Problems Committee)

Comment Let $\mathcal{E}$ be the circle in the problem statement centred at Q and passing through R. If we relax the condition that Q is the centre of $\mathcal{E}$ by requiring only that the centre of $\mathcal{E}$ lies on $\mathcal{C}$, then the conclusion of the problem is still true.

Here is a proof of this generalisation of the problem.



We have only considered the configuration where X lies between P and Y. Other configurations may be dealt with similarly.

Let O be the centre of $\mathcal{E}$. Since $PR = PY$ (radii of $\mathcal{D}$), and $OR = OY$ (radii of $\mathcal{C}$), it follows that $\triangle OPR \equiv \triangle OPY$ (SSS). Thus

$$\angle YPO = \angle OPR. \tag{1}$$

In circle $\mathcal{C}$, chords OX and OR have equal length. Hence they subtend equal angles at the point P on the circumference of this circle. Thus

$$\angle XPO = \angle OPR. \tag{2}$$

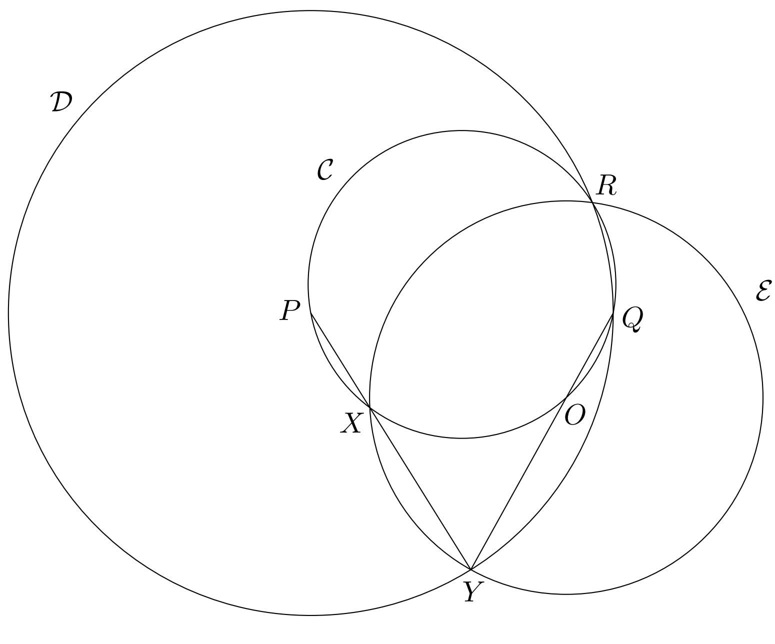
From (1) and (2) we have $\angle XPO = \angle YPO$. Thus P, X, and Y are collinear. $\square$

Comments (Angelo Di Pasquale, Director of Training, AMOC)

The astute reader may have noticed that the point Q did not get mentioned at all in the previous proof. The reason is because, in this generalisation, the point Q is irrelevant to the proof. Hence the diagram could have been drawn without the point Q and without the line segments PQ and QR. This minimal statement of the generalisation of the problem is as follows.

> Let P, O, and R be three points on a circle $\mathcal{C}$. Let $\mathcal{D}$ be a circle centred at P and passing through R, and let $\mathcal{E}$ be the circle centred at O and passing through R. Suppose that $\mathcal{E}$ intersects $\mathcal{C}$ again at X and $\mathcal{D}$ again at Y. Then P, X, and Y lie on a line.

In fact we can say even more! There is a combinatorial symmetry in the problem statement between $\mathcal{D}$ and $\mathcal{E}$ with respect to $\mathcal{C}$. Specifically, exchanging points P and O also exchanges $\mathcal{D}$ and $\mathcal{E}$. Thus if $\mathcal{D}$ intersects $\mathcal{C}$ again at Q, then O, Q, and Y also lie on a line.



7. **Solution 1** (Tianyue (Ellen) Zheng, year 12, Smith's Hill High School, NSW)

Since $b_1 > 1$, it follows inductively that $b_n > 1$ for all $n \geq 2$.

Observe that b_n is a perfect square for all $n \geq 2$.

For any perfect square m^2, all of the factors of m^2, except for m, come in pairs $(d, m^2/d)$ where $1 \leq d < m$. Hence square numbers have an odd number of factors. Thus b_n is an odd perfect square for all $n \geq 3$. Moreover since $b_n > 1$, it follows that $b_n \geq 9$ for all $n \geq 3$.

Consider any integer $n \geq 3$. Then $b_n = m^2$ for some odd integer $m \geq 3$. As explained in the previous paragraph, apart from m, the factors of m^2 come in pairs $(d, m^2/d)$ where $1 \leq d < m$. Since m^2 is odd, each such d is odd. The number of odd positive integers less than m is equal to $\frac{m-1}{2}$. It follows that the number of factors of m^2 is at most $2 \times \frac{m-1}{2} + 1 = m$. Therefore $b_{n+1} \leq b_n$.

However, for $b_{n+1} = b_n$ to occur, *all* of the odd positive integers less than m must be factors of m^2. In particular $m - 2 \mid m^2$. But

$$\gcd(m, m-2) = \gcd(m, 2) = 1 \quad \Rightarrow \quad \gcd(m^2, m-2) = 1.$$

Since $m - 2 \mid m^2$, it follows that $m - 2 = 1$. Thus $m = 3$, and $b_n = 9$. So from b_3 onward, the sequence is strictly decreasing until it reaches the fixed point $b_n = 9$. $\square$

Solution 2 (James Bang, year 11, Baulkham Hills High School, NSW)

As in solution 1 we deduce that each member of the sequence from b_3 onward is an odd perfect square that is greater than or equal to 9.

Consider any integer $n \geq 3$. As in solution 1 it suffices to prove that $b_{n+1} \leq b_n$ with equality if and only if $b_n = 9$.

Let $b_n = \prod_{i=1}^{r} p_i^{2a_i}$, where $p_1 < p_2 < \cdots < p_r$ are the prime factors of b_n.

The number of factors of b_n is given by $\prod_{i=1}^{r}(2a_i + 1)$. Hence $b_{n+1} = \prod_{i=1}^{r}(2a_i + 1)^2$.

It follows that $b_{n+1} \leq b_n$ if and only if

$$\prod_{i=1}^{r} p_i^{a_i} \geq \prod_{i=1}^{r}(2a_i + 1). \tag{1}$$

To finish the proof it suffices to show that

$$p^a \geq 2a + 1 \tag{2}$$

with equality if and only if $p = 3$ and $a = 1$.

Since $p^a > 3^a$ whenever $p > 3$, it suffices to prove (2) just for the case $p = 3$. We proceed by induction on a.

For $a = 1$, it is readily seen that inequality (2) is in fact an equality.

For the inductive step, suppose that $3^a \geq 2a + 1$ for some $a \geq 1$. It follows that

$$3^{a+1} \geq 3(2a + 1) > 2(a + 1) + 1.$$

This concludes the induction and the proof. $\square$

Comment Here are two more alternative ways of proving inequality (2).

Alternative 1 (Angelo Di Pasquale, Director of Training AMOC)

Using the binomial theorem, the inequality follows from

$$p^{2a} \geq (1 + 2)^{2a} \geq 1 + \binom{2a}{1} \cdot 2 + \binom{2a}{2} \cdot 2^2 = 8a^2 + 1 \geq (2a + 1)^2. \qquad \square$$

Alternative 2 (Ivan Guo, AMOC Senior Problems Committee)

Factoring the difference of perfect ath powers, the inequality follows from

$$p^a - 1 \geq 3^a - 1 = (3 - 1)(3^{a-1} + 3^{a-2} + \cdots + 1) \geq 2(1 + 1 + \cdots + 1) = 2a. \qquad \square$$

8. For ease of exposition, in each of the solutions that follow, whenever we say something has mass n, then we use this as shorthand for saying that it has mass n kilograms.

Solution 1 (James Bang, year 11, Baulkham Hills High School, NSW)

Answer 1009

Suppose that Amy has $r \geq 1010$ rocks of masses $m_1, m_2, \ldots, m_r$. For each integer i with $1 \leq i \leq r$, let

$$S_i = m_1 + m_2 + \cdots + m_i.$$

Note that $S_1, S_2, \ldots, S_r$ is a strictly increasing sequence of $r \geq 1010$ positive integers, each of which is in the range $1, 2, \ldots, 2018$.

Consider the 1009 pairs

$$(1, 1010), (2, 1011), (3, 1012), \ldots, (1009, 2018).$$

By the pigeonhole principle one of the above pairs contains two of the S_i. Suppose that $(t, t + 1009) = (S_i, S_j)$. Then $j > i$, and

$$1009 = S_j - S_i = m_j + m_{j-1} + \cdots + m_{i+1}.$$

Thus $m_j, m_{j-1}, \ldots, m_{i+1}$ have total mass equal to 1009. This shows that the answer cannot be greater than or equal to 1010.

To complete the proof, here is a construction for exactly 1009 rocks. Suppose that Amy has 1 rock of mass 1010, and 1008 rocks of mass 1. Then the pile that contains the rock of mass 1010 obviously has mass exceeding 1009. □

Comment 1 Some students found a somewhat geometric version of the above argument which basically goes as follows.

Suppose Amy has at least 1010 rocks. Consider a circle of circumference 2018 and partition it into arcs whose lengths correspond to the masses of the rocks. The endpoints of the arcs are 1010 (or more) of the vertices of a regular 2018-gon, and so by the pigeonhole principle there is a diametrically opposite pair of endpoints. □

Comment 2 Some students stated and proved the following fairly well-known lemma, and used this to complete the proof.

Lemma Any sequence of n integers contains a nonempty subsequence whose sum is a multiple of n.

Proof Let the integers be $m_1, m_2, \ldots, m_n$. Let $S_i = m_1 + m_2 + \cdots + m_i$ for $1 \leq i \leq n$.

If $S_i \equiv 0 \pmod{n}$ for some i, then we are done.

Otherwise, by the pigeonhole principle we have $S_i \equiv S_j \pmod{n}$ for some $i < j$. It follows that $S_j - S_i = m_{i+1} + m_{i+2} + \cdots + m_j \equiv 0 \pmod{n}$. □

To complete the proof of the given problem, if we have at least 1010 rocks, consider any 1009 of them. By the lemma, a nonempty sub-collection of those rocks has mass equal to a multiple of 1009. Since the total mass of the sub-collection is strictly less than 2018, it is equal to 1009. □

Solution 2 (William Steinberg, year 10, Scotch College, WA)

Suppose that Amy has at least 1010 rocks. Suppose that a of the rocks have unit mass and b of the rocks have mass greater than 1. If $b = 0$, then $a = 2018$, and in this case it is obvious that the rocks can be divided into two piles of equal mass. Hence $b > 0$. Let $M \geq 2$ be the mass of the heaviest rock. We have the following inequalities.

$$a + b \geq 1010 \tag{1}$$

$$a + 2(b - 1) + M \leq 2018 \tag{2}$$

Inequality (1) simply states that there are at least 1010 rocks. Inequality (2) is found by estimating the total mass. It is true because among the b rocks of non-unit mass, one of them has mass M, and the rest each have mass at least 2.

Inequality (1) implies $b \geq 1010 - a$. Substituting this into inequality (2) yields

$$2018 \geq M + a + 2(b - 1) \geq M + a + 2(1010 - a - 1) = M + 2018 - a.$$

It follows that $a \geq M$.

Next we create a pile of rocks with total mass 1009 as follows. First, continually choose rocks of non-unit mass and add them to the pile one at a time until one of the following two things occurs.

(i) It is not possible to add any more rocks of non-unit mass without the mass of the pile under construction exceeding 1009.

(ii) We run out of rocks of non-unit mass and the mass of the pile is at most 1009.

If (i) occurs, then the difference between the current mass of the pile and 1009 is less than M. Since $a \geq M$ we can top up the pile with rocks of unit mass until the pile has mass exactly 1009.

If (ii) occurs, then all remaining rocks have unit mass. Hence we can top up the pile with these until the pile has mass exactly 1009.

Either way a pile of mass 1009 has been created. This shows that the answer cannot be greater than or equal to 1010.

To complete the proof, here is a construction for exactly 1009 rocks. Suppose that Amy has 1009 rocks, each of mass 2. If some of these rocks are put into a pile, then the total mass of the pile will be even, and so cannot have mass 1009. □

Solution 3 (Guowen Zhang, year 12, St Joseph's College, QLD)

We claim the answer is 1009.

As in solution 1, if Amy has 1 rock of mass 1010 and 1008 rocks of mass 1, then it is impossible to divide the rocks into two piles of equal mass. Hence the answer is greater than or equal to 1009. The proof that this is the largest possible number of rocks that Amy could have follows from the following lemma.

Lemma Let n be any positive integer. Suppose that Amy has at least $n+1$ rocks where the mass of each rock is a positive integer, and the total mass of all the rocks is $2n$. Then it is always possible to divide the rocks into two piles, each of mass n.

Proof We proceed by induction on n.

For the case $n = 1$, Amy has two rocks with total mass 2. This implies that each of Amy's rocks has unit mass, from which the result immediately follows.

For the inductive step, let us assume that the lemma is true for $n = k$. Consider the case $n = k + 1$. Thus Amy has at least $k + 2$ rocks whose total mass is $2k + 2$.

If all of Amy's rocks have mass greater than or equal to 2, then the total mass of all the rocks is greater than or equal to $2(k + 2) > 2k + 2$, which is a contradiction. Hence at least one of Amy's rocks has unit mass. If all of the rocks have unit mass then the result follows immediately. Hence we may let the masses of the rocks be

$$1, m_1, m_2, \ldots, m_r \tag{*}$$

where $r \geq k + 1$ and $m_r > 1$.

Consider the situation where we have r rocks of masses

$$m_1, m_2, \ldots, m_{r-1}, m_r - 1.$$

Observe that the number of rocks above is greater than or equal to $k + 1$, and the total mass of these rocks is equal to $2k$. From the inductive assumption we can divide these into two piles, each of total mass k. Next change $m_r - 1$ to m_r and add a single rock of unit mass to the pile not containing the rock of mass $m_r - 1$. This gives a division of the rocks with masses given in (*) into two piles, each having mass $k + 1$. This concludes the induction, and the proof. $\square$

Comment It is possible to strengthen the lemma as follows.

Lemma Let n be any positive integer. Suppose that Amy has at least $n+1$ rocks where the mass of each rock is a positive integer, and the total mass of all the rocks is $2n$. Then for any integer N with $0 \leq N \leq 2n$ it is always possible to divide the rocks into two piles, one of which has mass N.

The proof is very similar to the one given above.

AUSTRALIAN MATHEMATICAL OLYMPIAD RESULTS

Name	School	Year
Perfect Score and Gold		
James Bang	Baulkham Hills High School NSW	11
Jack Gibney	Penleigh and Essendon Grammar School VIC	12
William Han	Macleans College NZ	12*
Grace He	Methodist Ladies' College VIC	10
Charles Li	Camberwell Grammar School VIC	12
Haobin (Jack) Liu	Brighton Grammar School VIC	12
Jerry Mao	Caulfield Grammar School, Wheelers Hill VIC	12
William Steinberg	Scotch College WA	10
Ethan Tan	Cranbrook School NSW	11
Fengshuo (Fredy) Ye (Yip)	Chatswood High School NSW	8
Guowen Zhang	St Joseph's College QLD	12
Gold		
William Hu	Christ Church Grammar School WA	12
Yasiru Jayasooriya	James Ruse Agricultural High School NSW	10
Hadyn Tang	Trinity Grammar School VIC	9
Ziqi Yuan	Narrabundah College ACT	11
Silver		
Andres Buritica	Scotch College VIC	9
Andrew Chen	St Kentigern College NZ	13*
Linus Cooper	James Ruse Agricultural High School NSW	12
Liam Coy	Sydney Grammar School NSW	10
Haowen Gao	Knox Grammar NSW	11
Sharvil Kesarwani	Merewether High School NSW	11
Johnathan Leung	King's College NZ	10*
Keiran James Lewellen	Te Aho o Te Kura Pounamu NZ	13*
Steven Lim	Hurlstone Agricultural High School NSW	12
Adrian Lo	Newington College NSW	10
Forbes Mailler	Canberra Grammar School ACT	12
Oliver Papillo	Camberwell Grammar School VIC	11
Preet Patel	Vermont Secondary College VIC	11
Tang (Michael) Sui	Caulfield Grammar School, Wheelers Hill VIC	12
Anthony Tew	Pembroke School SA	12
Xutong Wang	Auckland International College NZ	13*
Bronze		
Grace Chang	St Kentigern College NZ	11*
Kieren Connor	Sydney Grammar School NSW	12
Carl Gu	Melbourne High School VIC	12
Yifan Guo	Glen Waverley Secondary College VIC	12
Rick Han	Macleans College NZ	10*
Matthew Kerr	St Anthony's Catholic College QLD	12
David Lee	James Ruse Agricultural High School NSW	11

**NZ school year*

Name	School	Year
Alan Li	Lincoln High School NZ	12*
William Li	Barker College NSW	12
Zefeng (Jeff) Li	Caulfield Grammar School, Caulfield Campus VIC	11
Ishan Nath	John Paul College NZ	11*
Anthony Pisani	St Paul's Anglican Grammar VIC	11
Ken Gene Quah	Melbourne High School VIC	10
Marcus Rees	Hobart College TAS	11
Lachlan Rowe	Canberra College ACT	11
Mikhail Savkin	Gosford High School NSW	10
Albert Smith	Christ Church Grammar School WA	12
Ryan Stocks	Radford College ACT	12
Brian Su	James Ruse Agricultural High School NSW	12
Ruiqian Tong	Presbyterian Ladies' College VIC	12
Andrew Virgona	Smith's Hill High School NSW	12
Daniel Wiese	Scotch College WA	10
Kevin Wu	Scotch College VIC	11
Shine Wu	Newlands College NZ	13*
Zijin (Aaron) Xu	Caulfield Grammar School, Wheelers Hill VIC	10
Xinyue Alice Zhang	A. B. Paterson College QLD	12
Yang Zhang	St Joseph's College QLD	10
Linan (Frank) Zhao	Geelong Grammar School VIC	11
Tianyue (Ellen) Zheng	Smith's Hill High School NSW	12
Stanley Zhu	Melbourne Grammar School VIC	12
Honourable Mention		
Vincent Abbott	Hale School WA	12
Evgeniya Artemova	Presbyterian Ladies' College VIC	11
Huxley Berry	Perth Modern School WA	10
Junhua Chen	Caulfield Grammar School, Wheelers Hill VIC	10
Jonathan Chew	Christ Church Grammar School WA	11
Matthew Cho	St Joseph's College QLD	10
Shevanka Dias	All Saints' College WA	11
Christopher Do	Penleigh and Essendon Grammar School VIC	11
Vicky Feng	Methodist Ladies' College NSW	11
Eva Ge	James Ruse Agricultural High School NSW	9
Mikaela Gray	Brisbane State High School QLD	9
Ryan Gray	Brisbane State High School QLD	11
Dhruv Hariharan	Knox Grammar NSW	9
Remi Hart	All Saints' College WA	10
Tom Hauck	All Saints Anglican School QLD	10
Jocelin Hon	James Ruse Agricultural High School NSW	11
Claire Huang	Radford College ACT	10
Hollis Huang	Tintern Grammar School VIC	12
Shivasankaran Jayabalan	Rossmoyne Senior High School WA	12
Anagha Kanive-Hariharan	James Ruse Agricultural High School NSW	10

**NZ school year*

Name	School	Year
Reef Kitaeff	Perth Modern School WA	11
Samuel Lam	James Ruse Agricultural High School NSW	10
Andy Li	Presbyterian Ladies' College VIC	11
Jeffrey Li	North Sydney Boys High School NSW	11
Yueqi (Rose) Lin	Shenton College WA	12
Linda Lu	The Mac.Robertson Girls' High School VIC	11
Wenquan Lu	Barker College NSW	11
Nathaniel Masfen-Yan	King's College NZ	11*
Hilton Nguyen	Sydney Technical High School NSW	12
Angus Ritossa	St Peter's College SA	11
Ethan Ryoo	Knox Grammar NSW	9
Jianyi (Jason) Wang	Queensland Academy for Science, Mathematics and Technology QLD	10
Ziang (Tommy) Wei	Scotch College VIC	12
Emilie Wu	James Ruse Agricultural High School NSW	11
Lucinda Xiao	Methodist Ladies' College VIC	11
Jason Yang	James Ruse Agricultural High School NSW	11
Xinrong Yao	Auckland International College NZ	12*
Christine Ye	Qs School VIC	8
Elizabeth Yevdokimov	St Ursula's College QLD	9
Jasmine Zhang	Macleans College NZ	11*
Zirui (Harry) Zhang	Christian Brothers College VIC	9
Yufei (Phoebe) Zuo	Tara Anglican School for Girls NSW	12

**NZ school year*

AUSTRALIAN MATHEMATICAL OLYMPIAD STATISTICS

Score Distribution/Problem

Number of Students/Score	Problem Number							
	1	2	3	4	5	6	7	8
0	2	49	59	58	14	25	44	18
1	2	4	10	15	0	4	11	65
2	14	3	1	6	5	23	13	7
3	2	2	0	3	0	0	6	1
4	0	0	0	2	0	1	1	1
5	0	2	0	3	0	1	6	1
6	20	12	1	4	4	2	4	3
7	81	49	50	30	98	65	36	25
Average	**6**	**3.6**	**3**	**2.4**	**6**	**4.3**	**3**	**2.3**