AUSTRALIAN MATHS TRUST

Australian Mathematical Olympiad 2019

DAY 1
Tuesday, 5 February 2019
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Find all real numbers r for which there exists exactly one real number a such that when

$$(x+a)(x^2+rx+1)$$

is expanded to yield a cubic polynomial, all of its coefficients are greater than or equal to zero.

2. For each positive integer n, the nth *triangular number* is the sum of the first n positive integers. Let a, b, c be three consecutive triangular numbers with $a < b < c$.

Prove that if $a+b+c$ is a triangular number, then b is three times a triangular number.

3. Let A, B, C, D, E be five points in order on a circle $\mathcal{K}$. Suppose that $AB = CD$ and $BC = DE$. Let the chords AD and BE intersect at the point P.

Prove that the circumcentre of triangle AEP lies on $\mathcal{K}$.

4. Let Q be a point inside the convex polygon $P_1P_2\cdots P_{1000}$. For each $i = 1, 2, \ldots, 1000$, extend the line P_iQ until it meets the polygon again at a point X_i. Suppose that none of the points $X_1, X_2, \ldots, X_{1000}$ is a vertex of the polygon.

Prove that there is at least one side of the polygon that does not contain any of the points $X_1, X_2, \ldots, X_{1000}$.

The Mathematics/Informatics Olympiads are supported by the Australian Government through the National Innovation and Science Agenda.

AUSTRALIAN MATHS TRUST

Australian Mathematical Olympiad 2019

DAY 2
Wednesday, 6 February 2019
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. A *fancy triangle* is an equilateral triangular array of integers such that the sum of the three numbers in any unit equilateral triangle is a multiple of 3. For example,

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 2 & \\ 5 & & 7 & & 3 \end{array}$$

is a fancy triangle with three rows because the sum of the numbers in each of the following four unit equilateral triangles is a multiple of 3.

$$\begin{array}{ccc} & 1 & \\ 0 & & 2 \end{array} \qquad \begin{array}{ccc} & 0 & \\ 5 & & 7 \end{array} \qquad \begin{array}{ccc} 0 & & 2 \\ & 7 & \end{array} \qquad \begin{array}{ccc} & 2 & \\ 7 & & 3 \end{array}$$

Suppose that a fancy triangle has ten rows and that exactly n of the numbers in the triangle are multiples of 3.

Determine all possible values for n.

6. Let $\mathcal{K}$ be the circle passing through all four corners of a square $ABCD$. Let P be a point on the minor arc CD, different from C and D. The line AP meets the line BD at X and the line CP meets the line BD at Y. Let M be the midpoint of XY.

Prove that MP is tangent to $\mathcal{K}$.

7. Akshay writes a sequence $a_1, a_2, \ldots, a_{100}$ of integers in which the first and last terms are equal to 0. Except for the first and last terms, each term a_i is larger than the average of its neighbours a_{i-1} and a_{i+1}.

What is the smallest possible value for the term a_{19}?

8. Let $n = 16^{3^r} - 4^{3^r} + 1$ for some positive integer r.

Prove that $2^{n-1} - 1$ is divisible by n.