Australian Mathematical Olympiad 2019

DAY 1
Tuesday, 5 February 2019
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Find all real numbers r for which there exists exactly one real number a such that when

$$(x+a)(x^2+rx+1)$$

is expanded to yield a cubic polynomial, all of its coefficients are greater than or equal to zero.

2. For each positive integer n, the nth *triangular number* is the sum of the first n positive integers. Let a, b, c be three consecutive triangular numbers with $a < b < c$.

Prove that if $a+b+c$ is a triangular number, then b is three times a triangular number.

3. Let A, B, C, D, E be five points in order on a circle $\mathcal{K}$. Suppose that $AB = CD$ and $BC = DE$. Let the chords AD and BE intersect at the point P.

Prove that the circumcentre of triangle AEP lies on $\mathcal{K}$.

4. Let Q be a point inside the convex polygon $P_1P_2\cdots P_{1000}$. For each $i = 1, 2, \ldots, 1000$, extend the line P_iQ until it meets the polygon again at a point X_i. Suppose that none of the points $X_1, X_2, \ldots, X_{1000}$ is a vertex of the polygon.

Prove that there is at least one side of the polygon that does not contain any of the points $X_1, X_2, \ldots, X_{1000}$.

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Australian Mathematical Olympiad 2019

DAY 2
Wednesday, 6 February 2019
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. A *fancy triangle* is an equilateral triangular array of integers such that the sum of the three numbers in any unit equilateral triangle is a multiple of 3. For example,

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 2 & \\ 5 & & 7 & & 3 \end{array}$$

is a fancy triangle with three rows because the sum of the numbers in each of the following four unit equilateral triangles is a multiple of 3.

$$\begin{array}{ccc} & 1 & \\ 0 & & 2 \end{array} \qquad \begin{array}{ccc} & 0 & \\ 5 & & 7 \end{array} \qquad \begin{array}{ccc} 0 & & 2 \\ & 7 & \end{array} \qquad \begin{array}{ccc} & 2 & \\ 7 & & 3 \end{array}$$

Suppose that a fancy triangle has ten rows and that exactly n of the numbers in the triangle are multiples of 3.

Determine all possible values for n.

6. Let $\mathcal{K}$ be the circle passing through all four corners of a square $ABCD$. Let P be a point on the minor arc CD, different from C and D. The line AP meets the line BD at X and the line CP meets the line BD at Y. Let M be the midpoint of XY.

Prove that MP is tangent to $\mathcal{K}$.

7. Akshay writes a sequence $a_1, a_2, \ldots, a_{100}$ of integers in which the first and last terms are equal to 0. Except for the first and last terms, each term a_i is larger than the average of its neighbours a_{i-1} and a_{i+1}.

What is the smallest possible value for the term a_{19}?

8. Let $n = 16^{3^r} - 4^{3^r} + 1$ for some positive integer r.

Prove that $2^{n-1} - 1$ is divisible by n.

1. **Answer** $r = -1$

Solution 1 (Levi Pesin, year 9, Homeschooled, SA)

Expanding $(x + a)(x^2 + rx + 1)$ yields

$$x^3 + (a + r)x^2 + (1 + ar)x + a.$$

Hence we seek all real numbers r such that there is exactly one real number a satisfying the following system of inequalities.

$$a + r \geq 0 \tag{1}$$
$$ar \geq -1 \tag{2}$$
$$a \geq 0 \tag{3}$$

If $r \geq 0$, then any $a \geq 0$ satisfies the above inequalities. So this is no good.

If $r < 0$, let $r = -s$ for $s > 0$. The inequalities become

$$a \geq s \tag{1'}$$
$$a \leq \frac{1}{s} \tag{2'}$$
$$a \geq 0 \tag{3'}$$

Since inequality $(3')$ automatically follows from inequality $(1')$, we only need to be concerned with inequalities $(1')$ and $(2')$. These may be summarised as

$$s \leq a \leq \frac{1}{s}. \tag{4}$$

However, there is exactly one value of a satisfying (4) if and only $s = \frac{1}{s}$.

Since $s > 0$, this implies $s = 1$ (and $a = 1$), and so $r = -1$. $\square$

Solution 2 (Eva Ge, year 10, James Ruse Agricultural High School, NSW)

As in solution 1, we seek all real numbers r such that there is exactly one real number a satisfying the following system of inequalities.

$$a + r \geq 0 \tag{1}$$
$$ar \geq -1 \tag{2}$$
$$a \geq 0 \tag{3}$$

Case 1 $r < -1$

Then (1) implies $a > 1$ while (2) implies $a < 1$, which is a contradiction. So this case does not occur.

Case 2 $r = -1$

Then (1) implies $a \geq 1$, while (2) implies $a \leq 1$. So $a = 1$, and this also satisfies (3). Hence $r = -1$ is a solution.

Case 3 $-1 < r < 0$

We may write $r = -s$ where $0 < s < 1$. Thus (1), (2) and (3) become

$$a \geq s \tag{1'}$$
$$as \leq 1 \tag{2'}$$
$$a \geq 0 \tag{3'}$$

Since $s < 1$, any a satisfying $s < a < 1$ also satisfies (1′), (2′) and (3′). Since there are infinitely many such a, this case does not occur.

Case 4 $r \geq 0$

Any $a \geq 0$ satisfies (1), (2) and (3). So this case does not occur. $\square$

Solution 3 (Samuel Lam, year 11, James Ruse Agricultural High School, NSW)

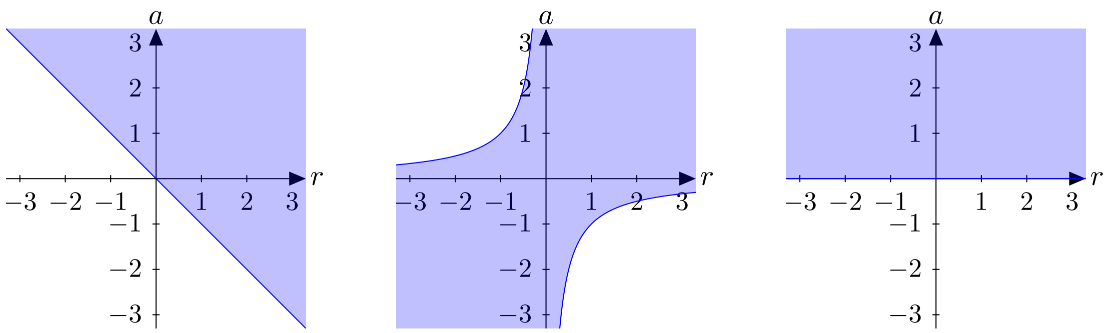
As in solution 1, we seek all real numbers r such that there is exactly one real number a satisfying the following system of inequalities.

$$a + r \geq 0 \tag{1}$$

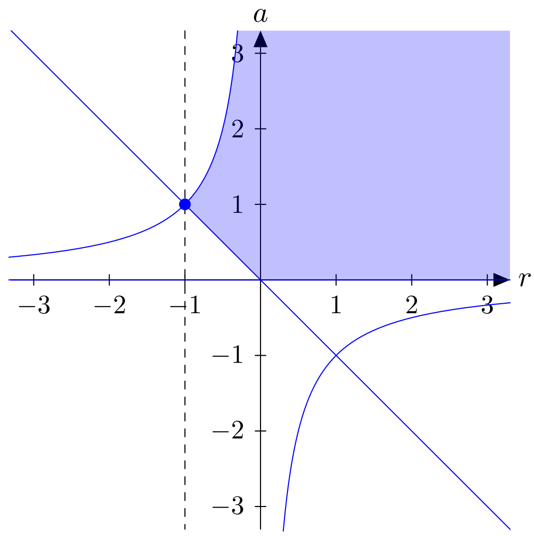
$$ar \geq -1 \tag{2}$$

$$a \geq 0 \tag{3}$$

We graph these three inequalities as follows. In each case the shaded region denotes the area defined by each inequality.



The intersection of the shaded regions denotes the region where all three inequalities are true.



For any vertical line passing through a value of r on the horizontal axis, each point on such a vertical line in the shaded area yields the corresponding values for a satisfying the required inequalities. Hence we seek the vertical lines which have exactly one point in common with the shaded area. The only such place is the indicated intersection point. This satisfies $a + r = 0$, $ar = -1$ and $a \geq 0$.

Substituting $r = -a$ into $ar = -1$ yields $a = \pm 1$. Since $a \geq 0$, we have $a = 1$. And since $r = -a$, it follows that $r = -1$. $\square$

2. **Solution** (Lucinda Xiao, year 12, Methodist Ladies' College, VIC)

The formula for the nth triangular number $T_n = 1 + 2 + \cdots + n$ is given by

$$T_n = \frac{n(n+1)}{2}. \tag{1}$$

Suppose that a, b, c are three consecutive triangular numbers, where $b = T_n$. Then $a = T_n - n$ and $c = T_n + n + 1$, and so

$$a + b + c = 3T_n + 1.$$

We are given that $a+b+c$ is equal to a triangular number, say T_m. Using formula (1), we have

$$\begin{aligned} \frac{3n(n+1)}{2} + 1 &= \frac{m(m+1)}{2} \\ \Leftrightarrow \qquad 3n(n+1) &= m^2 + m - 2 \\ &= (m-1)(m+2). \end{aligned} \tag{2}$$

Therefore $(m-1)(m+2)$ is a multiple of 3.

Since $m-1$ and $m+2$ differ by 3, it follows that $m-1$ is a multiple of 3 if and only if $m+2$ is too. Hence $m-1$ is a multiple of 3 and we may write $m-1 = 3k$ for some positive integer k. Putting this into (2) yields

$$\begin{aligned} 3n(n+1) &= 3k(3k+3) \\ \Leftrightarrow \qquad \frac{n(n+1)}{2} &= 3 \times \frac{k(k+1)}{2}. \end{aligned}$$

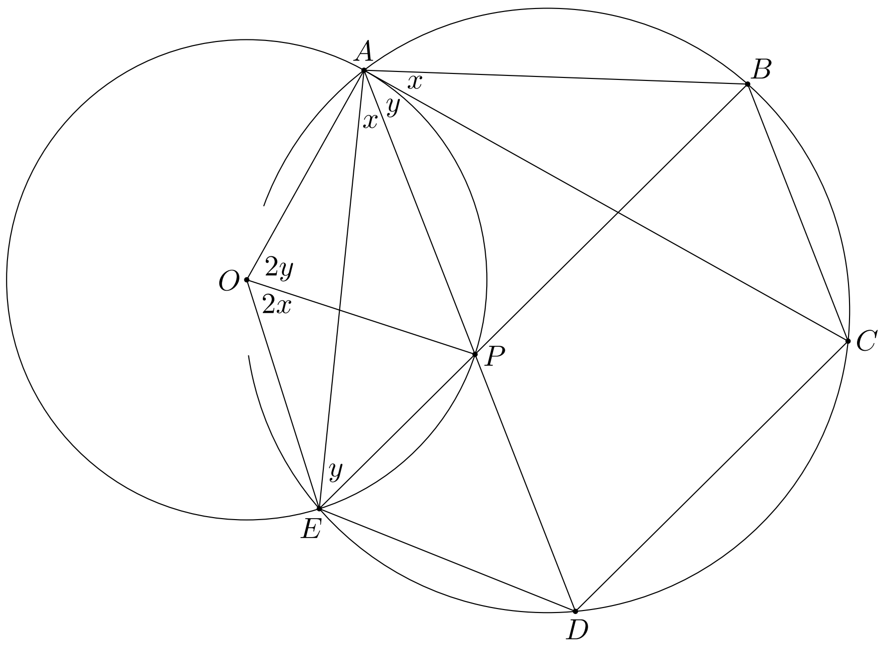
Thus $b = 3T_k$, as required. $\square$

3. **Solution 1** (Elizabeth Yevdokimov, year 10, St Ursula's College, QLD)

   Let O be the circumcentre of $\triangle AEP$. Let $x = \angle EAP$ and $y = \angle PEA$.

   Since the angle at the centre of a circle is twice the angle at the circumference, using the circle centred at O, we have $\angle EOP = 2x$ and $\angle POA = 2y$. It follows that

   $$\angle EOA = 2x + 2y. \tag{1}$$



   In a circle, equal chords subtend equal angles. We apply this twice to circle $\mathcal{K}$ as follows. Since $AB = CD$, we have $\angle DAC = \angle BEA = y$. And since $BC = DE$, we have $\angle CAB = \angle EAD = x$. It follows that $\angle EAB = 2x + y$. From the angle sum in $\triangle ABE$, we deduce that

   $$\angle ABE = 180^\circ - \angle EAB - \angle DEA = 180^\circ - 2x - 2y. \tag{2}$$

   Comparing (1) and (2), we see that $\angle EOA + \angle ABE = 180^\circ$. Therefore $ABEO$ is cyclic, and so O lies on $\mathcal{K}$. $\square$

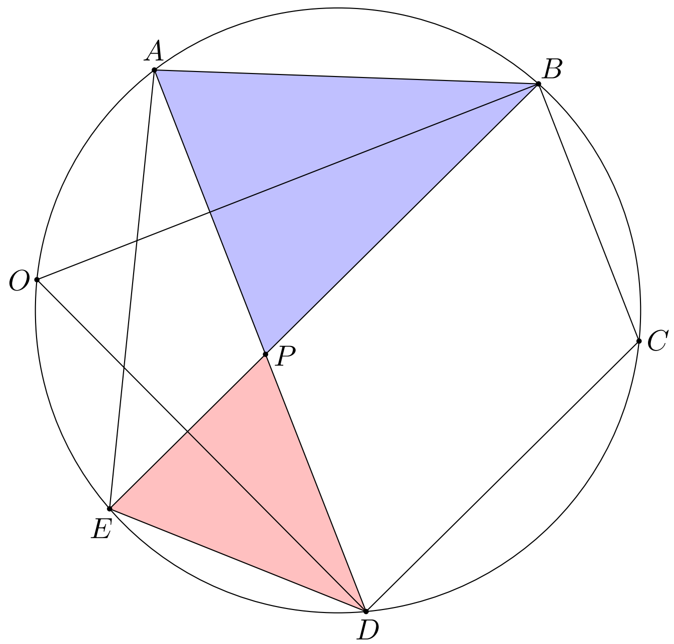
Solution 2 (Preet Patel, year 12, Vermont Secondary College, VIC)

Since $ABCD$ is cyclic with $AB = CD$, it follows that $ABCD$ is an isosceles trapezium with $AD \parallel BC$. Similarly $BCDE$ is an isosceles trapezium with $CD \parallel BE$. Consequently $BCDP$ is a parallelogram due to its opposite sides being parallel. Hence

$$BP = CD = BA \quad \text{and} \quad DP = CB = DE.$$

Hence triangles BAP and DEP are isosceles with bases PA and PE, respectively.

Let O be the midpoint of arc EA on $\mathcal{K}$ such that O lies on the opposite side of AE to points B, C and D. It follows that OB bisects $\angle ABE$, that is, $\angle ABP$.



Since $\triangle ABP$ is isosceles with base PA, we see that OB is also the perpendicular bisector of PA. Similarly, OD is the perpendicular bisector of PE. It follows that since O is the intersection of the perpendicular bisectors of two sides of $\triangle AEP$, it is the circumcentre of that triangle. $\square$

Solution 3 (Sharvil Kesarwani, year 12, Merewether High School, NSW)

Let O be the point on $\mathcal{K}$ such that CO is a diameter of $\mathcal{K}$. Note that $OA \perp AC$ and $OE \perp EC$.

In a circle, equal chords subtend equal angles. Since $CD = AB$, we have

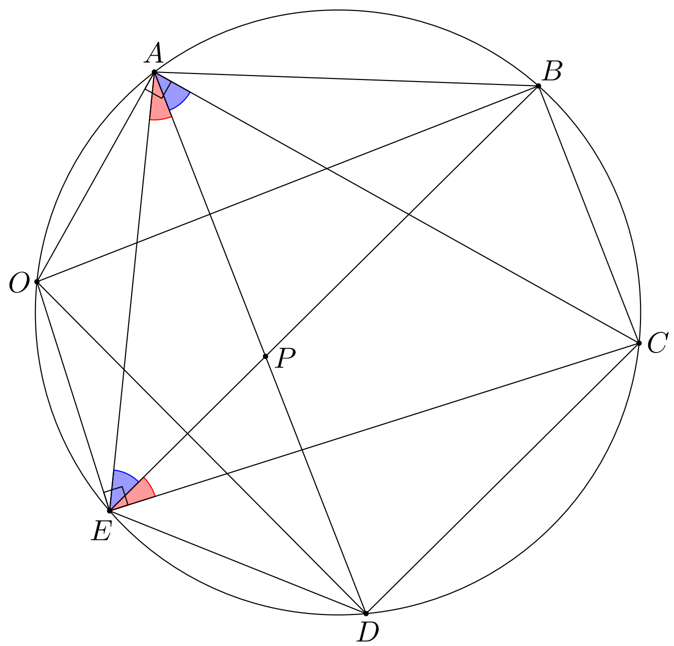
$$\angle PAC = \angle DAC = \angle BEA = \angle PEA.$$

By the alternate segment theorem, we see that CA is tangent to circle AEP at A.

Similarly, since $BC = DE$, we have

$$\angle CEP = \angle CEB = \angle EAB = \angle EAP$$

and so CE is tangent to circle AEP at E.

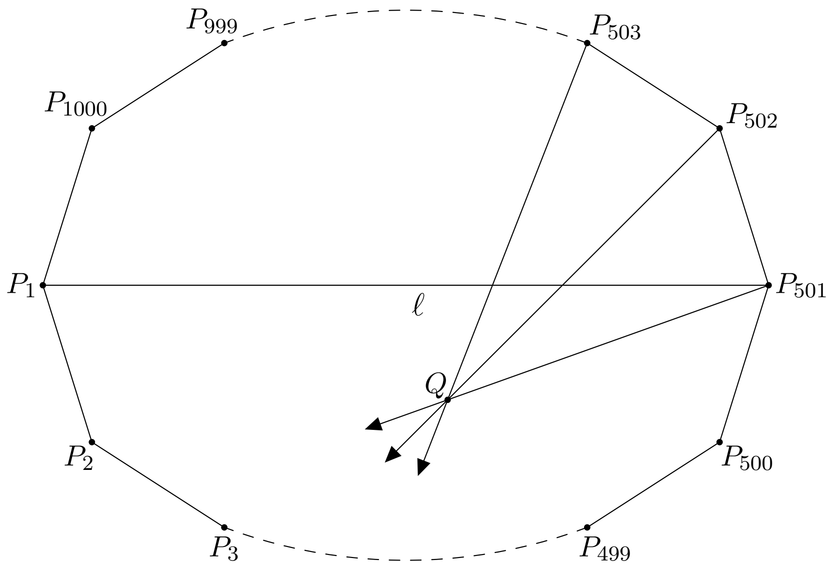


The centre of circle AEP is the intersection of the perpendicular to AC through A and the perpendicular to EC through C. But O is the intersection of these lines. Hence O is the centre of circle AEP. $\square$

4. **Solution 1** (Andres Buritica, year 10, Scotch College, VIC)

The diagonal $\ell = P_1P_{501}$ splits the polygon into two halves. Since Q does not lie on ℓ (otherwise $X_1 = P_{501}$ which is not permitted), it lies strictly inside one of these halves. Without loss of generality we may assume that Q lies inside $P_1P_2 \ldots P_{501}$.

Orient the original polygon so that ℓ is horizontal and so that $P_2, \ldots, P_{500}$ and Q lie below ℓ. Note that 500 edges of the polygon lie on each side of ℓ.



Each of the rays $\overrightarrow{P_{501}Q}, \overrightarrow{P_{502}Q}, \ldots, \overrightarrow{P_{1000}Q}, \overrightarrow{P_1Q}$ intersects ℓ. Therefore they intersect the polygon for a second time below ℓ. So the 501 points $X_{501}, X_{502}, \ldots, X_{1000}, X_1$ all lie below ℓ. Hence at most 499 of the X_i lie above ℓ. But 500 edges of the polygon lie above ℓ. So at least one of these edges does not contain an X_i, as required. $\square$

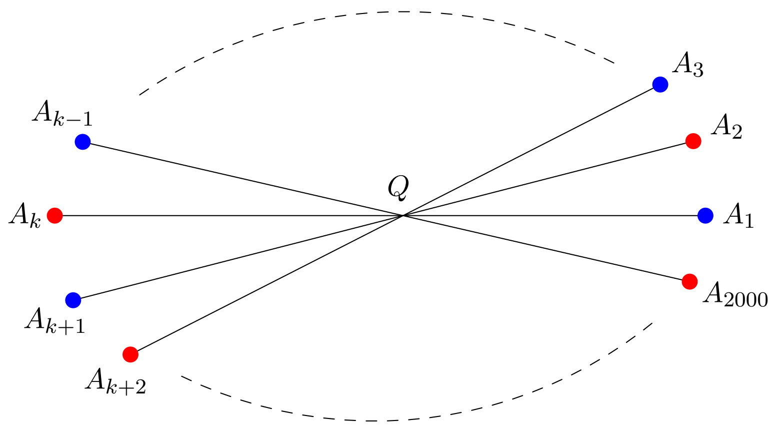
Solution 2 (Eva Ge, year 10, James Ruse Agricultural High School, NSW)

Colour each of the points P_i blue, each of the points X_i red, and each of the segments P_iX_i black. Hence there are 1000 blue points, 1000 red points, and 1000 black segments, and each black segment has one endpoint blue and one endpoint red. Also note that the 1000 black segments account for each of the coloured points exactly once.

Suppose, for the sake of contradiction that each side of the polygon contains a red point. Since there are exactly 1000 sides and 1000 red points, this implies that each side contains exactly one red point.

Let us relabel the coloured points as follows. Choose any blue point and relabel it as A_1. Then walk around the perimeter of the polygon labelling the coloured points in the order that they are encountered. In this way all the coloured points are labelled $A_1, A_2, \ldots, A_{2000}$. Moreover the blue points are $A_1, A_3, A_5, \ldots, A_{1999}$.

Let A_1A_k be the black segment containing A_1 as an endpoint. Since A_1 is blue, we know that A_k is red. This segment contains Q. Moreover all the other black segments also contain Q.

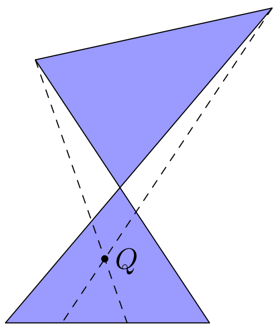


It follows that there are equally many A_i on each side of the line A_1A_k. Therefore $k = 1001$. But since 1001 is odd, A_{1001} is blue, which is a ontradiction. $\square$

Solution 3 (Frank Zhao, year 12, Geelong Grammar School, VIC)

Since there are 1000 points X_i and 1000 sides of the polygon, it suffices to prove that one side contains at least two of the X_i.

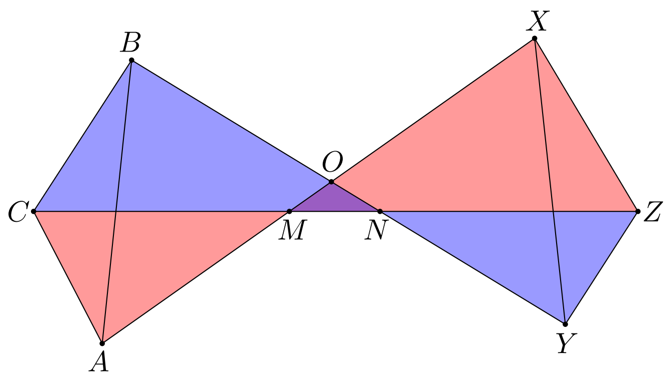
For any convex polygon with an even number of sides we define a *butterfly* to be the region formed by the two triangles cut out by a pair of consecutive main diagonals of the polygon. If Q lies inside a butterfly, then it is easy to see that the conclusion of the problem is true since a line that enters a triangle must exit it somewhere.



It suffices to prove that for any polygon with an even number of sides, the union of all its butterflies covers the entire polygon. We shall prove this by induction.

The base case of a convex quadrilateral $ABCD$ is trivial since the two butterflies $CABD$ and $CADB$ neatly cover the entire quadrilateral.

For the inductive step, suppose that for some even integer $n \geq 4$, every convex n-gon is contained in the union of its butterflies. Consider any convex polygon P with $n+2$ sides. Let CZ be a main diagonal of P. Consider the main diagonals AX and BY that are consecutive to CZ. So AC and CB are consecutive edges of the polygon, as are XZ and ZY. Also let AX meet BY at O. Orient P so that CZ is horizontal and O is on or above CZ. By relabelling $A \leftrightarrow B$ and $X \leftrightarrow Y$, if necessary, we may assume without loss of generality that B and X are also above CZ. Let CZ meet AX and BY at M and N, respectively. So with the given orientation of points, C, M, N, Z lie in that order on CZ.



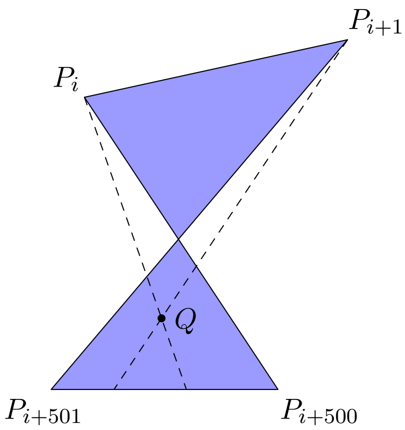
Consider the convex n-gon P' obtained from P by chopping off triangles ABC and XYZ. By the inductive assumption, the union of the butterflies of P' covers P'. Except for butterfly $XABY$, each butterfly of P' is also a butterfly of P. Replacing butterfly $XABY$ with butterflies $XACZ$ and $ZCBY$, we see that the butterflies of P completely cover P. This completes the induction and the proof. $\square$

Solution 4 (Angelo Di Pasquale, Director of Training, AMOC)

Define a *butterfly* as in solution 3. As in solution 3, it suffices to prove that Q lies inside a butterfly.

For any directed line $\overrightarrow{AB}$, we define its *positive* side to be the half-plane of points X such that $0 < \angle BAX < 180°$. We also define its *negative* side to be the half-plane of points X such that $180° < \angle BAX < 360°$. In both cases, the angle is directed anticlockwise modulo $360°$.

Without loss of generality, suppose that Q lies on the positive side of the directed line $\overrightarrow{P_1P_{501}}$, where we consider all subscripts modulo 1000. Then Q lies on the negative side of the directed line $\overrightarrow{P_{501}P_1}$. Hence, there exists an integer i with $1 \leq i \leq 500$ such that Q lies on the positive side of $\overrightarrow{P_iP_{i+500}}$, but on the negative side of $\overrightarrow{P_{i+1}P_{i+501}}$. Thus, Q lies inside the butterfly defined by P_iP_{i+500} and $P_{i+1}P_{i+501}$. $\square$



5. **Answer** $n = 0$, 18, 19 and 55

Solution (Mikaela Gray, year 10, Brisbane State High School, QLD)

Consider the four numbers in any two unit equilateral triangles that share a common edge as shown in the diagram.

$$\begin{array}{ccc} & a & \\ b & & c \\ & d & \end{array}$$

Since $a+b+c \equiv 0 \equiv b+c+d \pmod 3$, it follows that $d \equiv a \pmod 3$. Starting with the top two rows as being given by

$$\begin{array}{ccc} & a & \\ b & & c \end{array}$$

this implies that if we reduce the entries of the entire triangle modulo 3, it takes the following form.

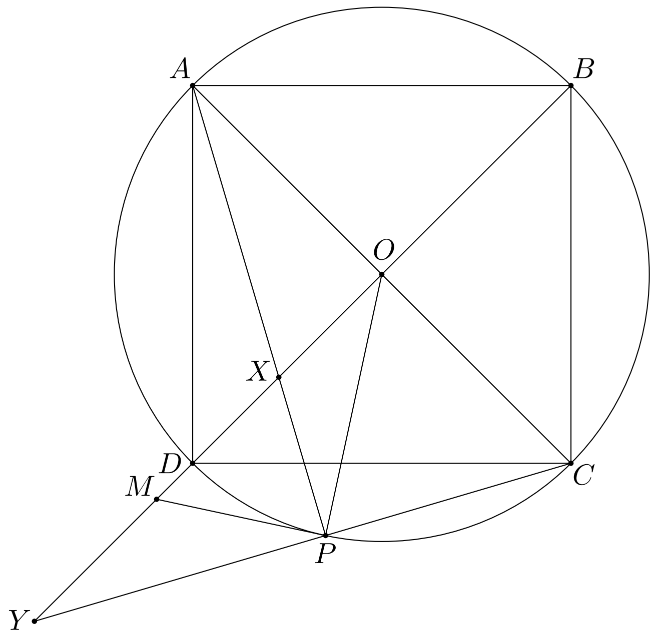
$$\begin{array}{ccccccccccccccccccc}
 & & & & & & & & & a & & & & & & & & & \\
 & & & & & & & & b & & c & & & & & & & & \\
 & & & & & & & c & & a & & b & & & & & & & \\
 & & & & & & a & & b & & c & & a & & & & & & \\
 & & & & & b & & c & & a & & b & & c & & & & & \\
 & & & & c & & a & & b & & c & & a & & b & & & & \\
 & & & a & & b & & c & & a & & b & & c & & a & & & \\
 & & b & & c & & a & & b & & c & & a & & b & & c & & \\
 & c & & a & & b & & c & & a & & b & & c & & a & & b & \\
a & & b & & c & & a & & b & & c & & a & & b & & c & & a
\end{array}$$

Note that a, b and c occur 19, 18 and 18 times, respectively, in the triangle. Also since $a+b+c \equiv 0 \pmod 3$, we see that either $a = b = c$, or a, b, c are equal to 0,1,2 in some order. The following table exhibits all the possibilities along with the corresponding values of n.

a	b	c	n
0	0	0	55
1	1	1	0
2	2	2	0
0	1	2	19
0	2	1	19
1	0	2	18
1	2	0	18
2	0	1	18
2	1	0	18

Hence the answers are $n = 0$, 18, 19 and 55. $\square$

6. **Solution 1** (Grace He, year 11, Methodist Ladies' College, VIC)



Since O is the centre of the circle, we may let $\angle OPA = \angle PAO = x$.

Since the diagonals of a square are perpendicular, using the angle sum in $\triangle OAX$ and the fact that vertically opposite angles are equal, we deduce that

$$\angle MXP = \angle OXA = 90^\circ - x.$$

Since M is the midpoint of XY and $\angle XPY = 90^\circ$, it follows that M is the circumcentre of $\triangle XYP$. Thus

$$MX = MP = MY.$$

Therefore

$$\angle XPM = \angle MXP = 90^\circ - x.$$

Consequently

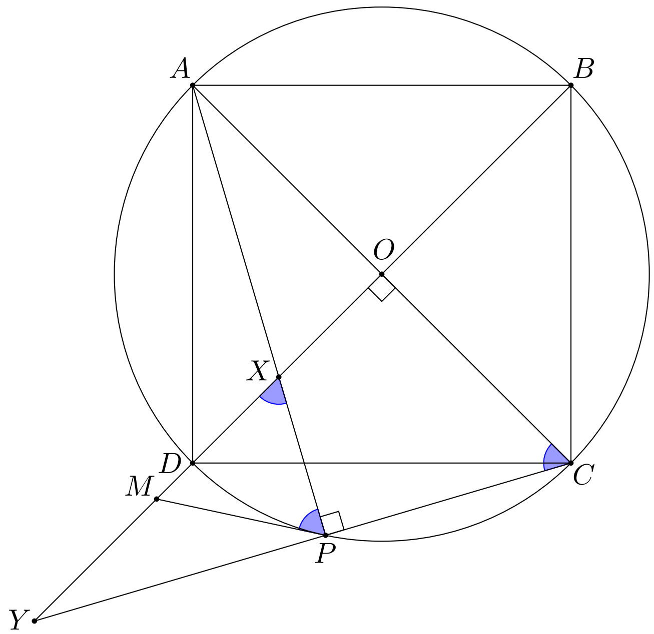
$$\angle OPM = \angle OPA + \angle XPM = x + (90^\circ - x) = 90^\circ.$$

Thus MP is perpendicular to the radius OP from which it follows that MP is tangent to the circle at P. $\square$

Solution 2 (Claire Huang, year 11, Radford College, ACT)

Our plan is to prove that $\angle APM = \angle ACP$ because then the result will follow by the alternate segment theorem.

Since diagonals of a square are perpendicular, we have $\angle XOC = 90°$. Also the diagonals of a square intersect at the square's circumcentre. Thus AC is a diameter of circle $ABCD$, and so $\angle CPA = 90°$. It follows that $XOCP$ is cyclic because $\angle XOC = \angle CPX = 90°$.



Next we observe that since $\angle XPY = 90°$, it follows XY is the diameter of the circle through X, P and Y. Hence M is the centre of this circle, and so

$$MX = MP = MY.$$

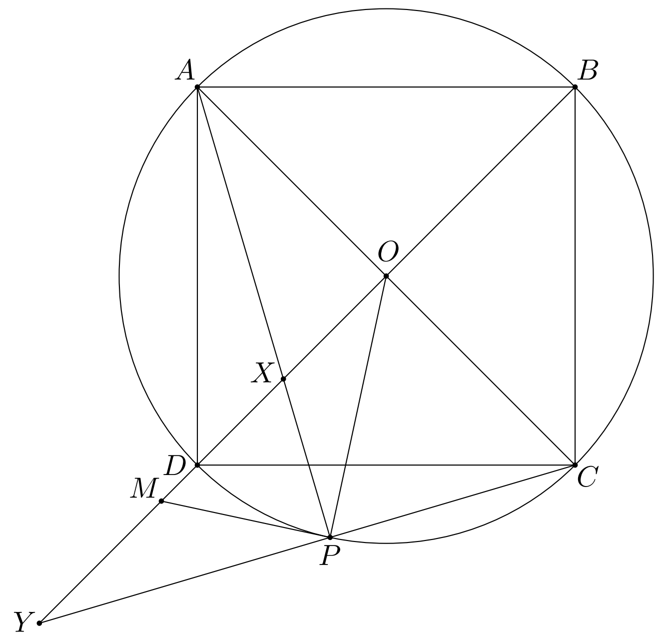
It follows that

$$\begin{aligned}\angle APM &= \angle XPM \\ &= \angle MXP \quad (MP = MX) \\ &= \angle OCP \quad (XOCP \text{ cyclic}) \\ &= \angle ACP\end{aligned}$$

as desired. $\square$

Solution 3 (Hadyn Tang, year 10, Trinity Grammar School, VIC)

The cyclic quadrilateral $ABCD$ is harmonic because $AB \cdot CD = BC \cdot DA$. From properties of harmonic quadrilaterals, the lines PA, PB, PC, PD form a harmonic bundle. Hence the intersections of these lines with any other line result in four harmonic points. In particular, intersecting PA, PB, PC, PD with the line BY yields that B, X, D, Y are harmonic.



Since M is the midpoint of XY and B, X, D, T are harmonic, a well-known calculation implies that

$$MX \cdot MY = MD \cdot MB. \tag{1}$$

Just the same, we show how to derive (1). Let $MD = a$, $DX = b$ and $XB = c$. Since $MX = MY$, we have $MY = a + b$. Since B, X, D, Y are harmonic in that order we have

$$\begin{aligned}
& DX \cdot YB = YD \cdot XB \\
\Leftrightarrow \quad & b(2a + 2b + c) = c(2a + b) \\
\Leftrightarrow \quad & 2ab + 2b^2 + bc = 2ac + bc \\
\Leftrightarrow \quad & ab + b^2 = ac \\
\Leftrightarrow \quad & a^2 + 2ab + b^2 = a^2 + ab + ac \\
\Leftrightarrow \quad & (a + b)^2 = a(a + b + c) \\
\Leftrightarrow \quad & MX \cdot MY = MD \cdot MB
\end{aligned}$$

as claimed.

Since $MP = MX = MY$ it follows that

$$MP^2 = MD \cdot MB$$

from which it follows that MP is tangent to $\mathcal{K}$ at P. $\square$

7. **Answer** 729

Solution 1 (James Bang, year 12, Baulkham Hills High School, NSW)

Since the a_i are integers, we have $a_n > \frac{1}{2}(a_{n-1} + a_{n+1})$ if and only if

$$a_n \geq \frac{a_{n-1} + a_{n+1} + 1}{2}. \tag{1}$$

From this, since $a_1 = 0$, we see that $a_2 \geq \frac{1}{2} + \frac{a_3}{2}$.

Using this and (1), we find $a_3 \geq \frac{a_2+a_4+1}{2} \geq \frac{\frac{1}{2}+\frac{a_3}{2}+a_4+1}{2}$, which implies $a_3 \geq 1 + \frac{2a_4}{3}$.

Similarly we have $a_4 \geq \frac{a_3+a_5+1}{2} \geq \frac{1+\frac{2a_4}{3}+a_5+1}{2}$, which implies $a_4 \geq \frac{3}{2} + \frac{3a_5}{4}$.

This leads to the following lemma, which we prove by induction.

Lemma 1 For $n = 1, 2, \dots, 99$ we have

$$a_n \geq \frac{n-1}{2} + \frac{(n-1)a_{n+1}}{n}.$$

Proof The base case $n = 1$ is trivially true since $a_1 = 0$.

For the inductive step, suppose that $a_n \geq \frac{n-1}{2} + \frac{(n-1)a_{n+1}}{n}$ for some positive integer $n \leq 98$. Then using (1) and the inductive assumption, we have

$$a_{n+1} \geq \frac{a_n + a_{n+2} + 1}{2} \geq \frac{\frac{n-1}{2} + \frac{(n-1)a_{n+1}}{n} + a_{n+2} + 1}{2} = \frac{n+1}{4} + \frac{(n-1)a_{n+1}}{2n} + \frac{a_{n+2}}{2}.$$

It follows that

$$\frac{(n+1)a_{n+1}}{2n} \geq \frac{n+1}{4} + \frac{a_{n+2}}{2} \quad \Rightarrow \quad a_{n+1} \geq \frac{n}{2} + \frac{na_{n+2}}{n+1}.$$

This completes the inductive step and the proof of lemma 1. □

Next, since $a_{100} = 0$, using lemma 1 for $n = 99$, we have $a_{99} \geq 49$.

Using this and lemma 1 for $n = 98$, we deduce $a_{98} \geq \frac{97}{2} + \frac{97 \cdot 49}{98} = 97$.

Applying the same process for $n = 97$ and 96, we deduce $a_{97} \geq 48 \cdot 3 = \frac{96 \cdot 3}{2}$ and $a_{96} \geq 95 \cdot 2 = \frac{95 \cdot 4}{2}$. This leads to the following lemma, which we prove by induction.

Lemma 2 For $n = 99, 98, \dots, 1$ we have

$$a_n \geq \frac{(n-1)(100-n)}{2}.$$

Proof The base case $n = 99$ has already been done. For the inductive step, suppose that $a_n \geq \frac{(n-1)(100-n)}{2}$ for some integer n with $2 \leq n \leq 99$. Then using lemma 1 and the inductive assumption we have

$$a_{n-1} \geq \frac{n-2}{2} + \frac{(n-2)a_n}{n-1} \geq \frac{n-2}{2} + \frac{(n-2) \cdot \frac{(n-1)(100-n)}{2}}{n-1} = \frac{(n-2)(101-n)}{2},$$

which completes the inductive step and the proof of lemma 2. □

Lemma 2 yields $a_{19} \geq \frac{18 \cdot 81}{2} = 729$. To complete the proof it suffices to show that $a_n = \frac{(n-1)(100-n)}{2}$ is a valid sequence. We need to verify that $a_{n+1} > \frac{a_n + a_{n+2}}{2}$, that is,

$$\frac{n(99-n)}{2} > \frac{\frac{(n-1)(100-n)}{2} + \frac{(n+1)(98-n)}{2}}{2}.$$

But the RHS of the above is equal to $\frac{n(99-n)-1}{2}$, and so the proof is complete. □

Solution 2 (Zefeng (Jeff) Li, year 12, Caulfield Grammar School, Caulfield, VIC)

Observe that

$$a_i > \frac{a_{i-1} + a_{i+1}}{2} \quad \Leftrightarrow \quad a_{i+1} - a_i < a_i - a_{i-1}.$$

With this in mind, let $d_i = a_{i+1} - a_i$ for $i = 1, 2, \ldots, 99$. We shall reformulate the problem in terms of the d_i.

The condition $a_i > \frac{1}{2}(a_{i-1} + a_{i+1})$ is equivalent to

$$d_1 > d_2 > \cdots > d_{99}. \tag{1}$$

For each non-negative integer n with $n \le 99$ we have

$$\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} (a_{i+1} - a_i) = a_{n+1} - a_1 = a_{n+1}. \tag{2}$$

Consequently the condition $a_1 = 0$ is already accounted for because LHS(2) is an empty sum when $n = 0$. The condition $a_{100} = 0$ is accounted for by

$$\sum_{i=1}^{99} d_i = 0. \tag{3}$$

The problem may now be reformulated as follows.

> Suppose that $d_1 > d_2 > \cdots > d_{99}$ are integers whose sum is equal to zero. Find the smallest possible value of $d_1 + d_2 + \cdots + d_{18}$.

Case 1 $d_{18} \le 31$

From (1) we have $d_{19} \le 30$, $d_{20} \le 29$, $\ldots$, $d_{99} \le -50$. So from (3) we deduce

$$a_{19} = \sum_{i=1}^{18} d_i = -\sum_{i=19}^{99} d_i \ge -\sum_{i=30}^{-50} i = -\frac{(30 + -50)(81)}{2} = 810.$$

Case 2 $d_{18} \ge 32$

Let $d = 32 + e$ where e is a non-negative integer. From (1) we have $d_{17} \ge 33 + e$, $d_{16} \ge 34 + e$, $\ldots$, $d_1 \ge 49 + e$. Therefore

$$a_{19} = \sum_{i=1}^{18} d_i \ge \sum_{i=32}^{49} i + e = \frac{(32 + 49)(18)}{2} + 18e = 729 + 18e \ge 729.$$

Thus in all cases $a_{19} \ge 729$.

Finally we observe that 729 is attainable when $d_i = 50 - i$ for $i = 1, 2, \ldots, 99$. $\square$

Comment

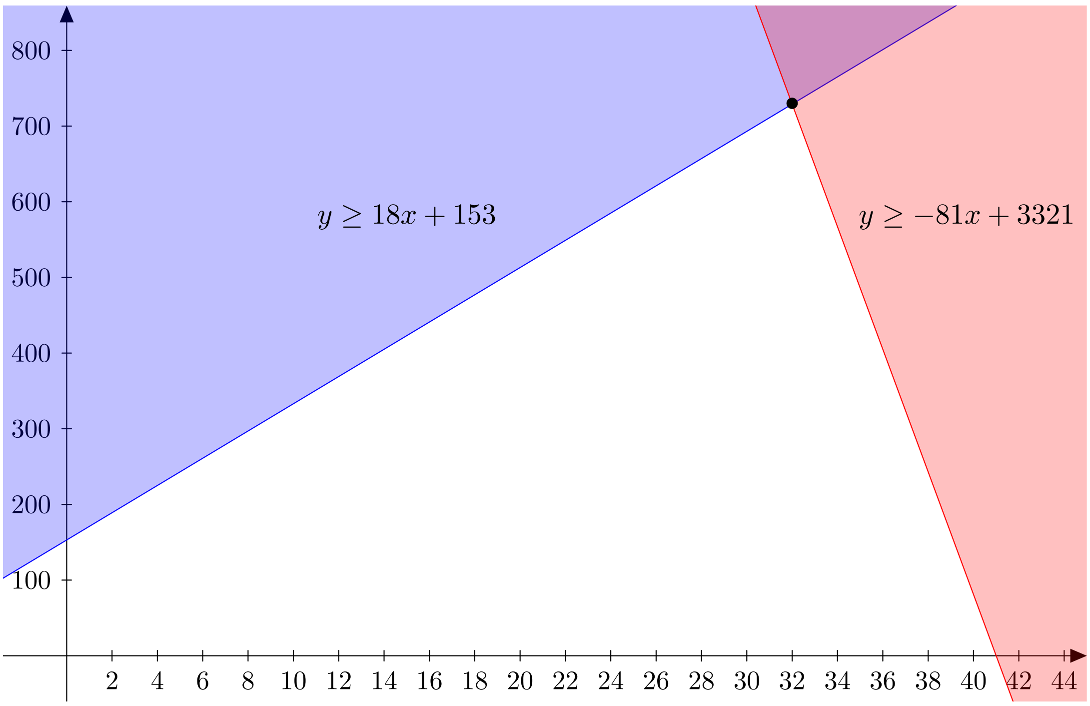
How might one determine that $d_{18} = 32$ is the critical value? One way is as follows. With $d_{18} = d$ we have $d_{19} \le d - 1$, $d_{20} \le d - 2$, $\ldots$, $d_{99} \le d - 81$. Hence

$$a_{19} = \sum_{i=1}^{18} d_i = -\sum_{i=19}^{99} d_i \ge -(d-1) - (d-2) - \cdots - (d-81) = -81d + 3321.$$

We also have $d_{17} \ge d + 1$, $d_{16} \ge d + 2$, $\ldots$, $d_1 \ge d + 17$ so that

$$a_{19} = \sum_{i=1}^{18} d_i \ge d + (d+1) + (d+2) + \cdots + (d+17) = 18d + 153.$$

The place where the minimum value of a_{19} occurs is at the intersection of the two lines $y = -81x + 3321$ and $y = 18x + 153$. It is a simple matter to compute that the the intersection point is $(32, 729)$, which corresponds to $d = 32$ and $a_{19} = 729$.



Solution 3 (Hadyn Tang, year 10, Trinity Grammar School, VIC)

Step 1 Define the sequences Δ_i and b_i.

Define the sequence $\Delta_1, \Delta_2, \ldots, \Delta_{99}$ by the formula $\Delta_i = 50 - i$. Then let $b_1 = 0$ and $b_{i+1} = b_i + \Delta_i$ for $i = 1, 2, \ldots, 99$. Note that

$$b_{100} = b_1 + \Delta_1 + \Delta_2 + \cdots + \Delta_{99} = 0 + 49 + 48 + \cdots + (-47) + (-48) + (-49) = 0.$$

Step 2 Show that $a_{19} = 729$ is possible using the b_i.

Observe that if $a_i = b_i$ for $i = 1, 2, \ldots, 100$, then

$$\frac{a_{i-1} + a_{i+1}}{2} < a_i \quad \Leftrightarrow \quad a_{i+1} - a_i < a_i - a_{i-1} \quad \Leftrightarrow \quad \Delta_i < \Delta_{i-1}$$

which is clearly true. Moreover $a_1 = a_{100} = 0$ and

$$a_{19} = b_{19} = 0 + 49 + 48 + \cdots + 32 = \frac{(18)(32 + 49)}{2} = 729.$$

Step 3 Show that $a_{19} < 729$ is impossible by showing that $a_i \geq b_i$ for all i.

Let $c_i = a_i - b_i$ for $i = 1, 2, \ldots, 100$. Note that $c_1 = c_{100} = 0$. Also observe that

$$\begin{aligned}
& \tfrac{1}{2}(a_{i-1} + a_{i+1}) < a_i \\
\Leftrightarrow \quad & a_{i+1} - a_i < a_i - a_{i-1} \\
\Leftrightarrow \quad & b_{i+1} + c_{i+1} - b_i - c_i < b_i + c_i - b_{i-1} - c_{i-1} \\
\Leftrightarrow \quad & c_{i+1} - c_i + \Delta_i < c_i - c_{i-1} + \Delta_{i-1} \\
\Leftrightarrow \quad & c_i - c_{i-1} + 1 < c_{i+1} - c_i \\
\Leftrightarrow \quad & c_{i+1} - c_i \leq c_i - c_{i-1} \qquad (1)
\end{aligned}$$

because the c_i are integers.

For each i, inequality (1) inductively implies that

$$c_k - c_{k-1} \leq c_i - c_{i-1} \quad \text{for all } k \geq i. \qquad (2)$$

Suppose, for the sake of contradiction, that at least one of the c_i is negative. Let i be the smallest positive integer such that $c_i < 0$. Note that $i \geq 2$ because $c_1 = 0$. Hence $c_{i-1} \geq 0$ and so $c_i - c_{i-1} < 0$. It follows from (2) that $c_k - c_{k-1} < 0$ for all $k \geq i$. Consequently $c_{100} < c_{99} < \cdots < c_i < 0$, which contradicts $c_{100} = 0$.

Hence we have shown that $c_i \geq 0$ for each i. Thus $a_i = b_i + c_i \geq b_i$ for each i. In particular we have $a_{19} \geq b_{19} = 729$. $\square$

Comment

It is straightforward to compute that

$$b_n = \sum_{i=1}^{n-1} \Delta_i = \sum_{i=1}^{n-1} (50 - i) = \frac{(50 - 1 + 50 - (n-1))(n-1)}{2} = \frac{(n-1)(100-n)}{2}$$

for each positive integer $n \leq 100$.

The above proof shows that for each i, the sequence $a_n = b_n$ is the unique sequence that minimises a_i. Thus the sequence that minimises any particular one of the a_i just happens to minimise all of the a_i simultaneously.

8. **Solution 1** (James Bang, year 12, Baulkham Hills High School, NSW)

For any positive integer x we have

$$x^2 - x + 1 \mid x^3 + 1 \mid x^6 - 1.$$[1]

Applying this for $x = 4^{3^r}$, since $n = x^2 - x + 1$, we have

$$n \mid (4^{3^r})^6 - 1 \quad \Leftrightarrow \quad n \mid 2^{4\cdot 3^{r+1}} - 1.$$

Thus it suffices to show that

$$2^{4\cdot 3^{r+1}} - 1 \mid 2^{n-1} - 1. \tag{1}$$

For any positive integers y, a, b with $a \mid b$, we have

$$y^a - 1 \mid y^b - 1.$$[2]

Applying this to (1), since $n - 1 = 4^{3^r}(4^{3^r} - 1)$, it suffices to show that

$$4 \cdot 3^{r+1} \mid 4^{3^r}(4^{3^r} - 1).$$

Clearly $4 \mid 4^{3^r}$ so it suffices to show that $3^{r+1} \mid 4^{3^r} - 1 = 2^{2\cdot 3^r} - 1$.

From Euler's theorem we have

$$2^{\varphi(3^{r+1})} \equiv 1 \pmod{3^{r+1}}$$

where φ is Euler's totient function.

Since $\varphi(3^{r+1}) = 2 \cdot 3^r$, it follows that $3^{r+1} \mid 2^{2\cdot 3^r} - 1$, as desired. $\square$

[1]This is because $x^6 - 1 = (x^3 + 1)(x^3 - 1)$ and $x^3 + 1 = (x + 1)(x^2 - x + 1)$.

[2]This is because if we write $b = ac$ and $z = y^a$, then $z - 1 \mid z^c - 1$ as $z^c - 1 = (z - 1)(z^{c-1} + z^{c-2} + \cdots + 1)$.

Solution 2 (Grace He, year 11, Methodist Ladies' College, VIC)

Since $n = 2^{4\cdot 3^r} - 2^{2\cdot 3^r} + 1$ is odd, for any positive integer $m \geq 12 \cdot 3^r$ we have

$$\begin{aligned}
2^m &\equiv 1 && \pmod n \\
&\equiv 2^{2\cdot 3^r} - 2^{4\cdot 3^r} && \pmod n \\
\Leftrightarrow \quad 2^{m-2\cdot 3^r} &\equiv 1 - 2^{2\cdot 3^r} && \pmod n \\
&\equiv -2^{4\cdot 3^r} && \pmod n \\
\Leftrightarrow \quad 2^{m-6\cdot 3^r} &\equiv -1 && \pmod n \\
&\equiv -2^{2\cdot 3^r} + 2^{4\cdot 3^r} && \pmod n \\
\Leftrightarrow \quad 2^{m-8\cdot 3^r} &\equiv -1 + 2^{2\cdot 3^r} && \pmod n \\
&\equiv 2^{4\cdot 3^r} && \pmod n \\
\Leftrightarrow \quad 2^{m-12\cdot 3^r} &\equiv 1 && \pmod n.
\end{aligned}$$

Applying the above inductively, we see that $n \mid 2^m - 1$ if and only if $n \mid 2^{m-k\cdot 12\cdot 3^r}$ where k is any integer such that $m \geq k \cdot 12 \cdot 3^r$. In particular if $12 \cdot 3^r \mid m$, then such a k exists, and this would imply that $n \mid 2^m - 1$.

We apply this to the case $m = n - 1 = 4^{3^r}(4^{3^r} - 1)$. Thus it suffices to show that $12 \cdot 3^r \mid 4^{3^r}(4^{3^r} - 1)$. Clearly $4 \mid 4^{3^r}$, therefore it only remains to show that $3^{r+1} \mid 4^{3^r} - 1$. We proceed by induction on r.

By inspection, the base case $r = 0$ is true.

For the inductive step, suppose that $3^{r+1} \mid 4^{3^r} - 1$ for some non-negative integer r. Then using the factorisation for the difference of two perfect cubes, we have

$$4^{3^{r+1}} - 1 = (4^{3^r} - 1)((4^{3^r})^2 + 4^{3^r} + 1).$$

By the inductive assumption, the first factor on the RHS is divisible by 3^{r+1}. So it suffices to show that the second factor is divisible by 3. However this is true because

$$(4^{3^r})^2 + 4^{3^r} + 1 \equiv (1^{3^r})^2 + 1^{3^r} + 1 = 3 \equiv 0 \pmod 3. \qquad \square$$

2019 AUSTRALIAN MATHEMATICAL OLYMPIAD STATISTICS

Score Distribution/Problem

Number of Students/Score	Problem Number							
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
0	13	17	38	64	5	27	35	108
1	4	33	3	15	8	6	36	5
2	3	7	4	2	3	6	25	0
3	6	0	0	1	2	2	13	1
4	3	0	4	1	3	0	2	0
5	4	0	4	3	4	0	1	0
6	10	3	8	0	10	3	3	1
7	88	71	70	45	96	87	16	16
Average	**5.6**	**4.3**	**4.5**	**2.7**	**6.0**	**5.0**	**2.0**	**1.0**