AUSTRALIAN MATHS TRUST

2020 Australian Mathematical Olympiad

DAY 1
Tuesday, 4 February 2020
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Determine all pairs (a, b) of non-negative integers such that

$$\frac{a+b}{2} - \sqrt{ab} = 1.$$

2. Amy and Bec play the following game. Initially, there are three piles, each containing 2020 stones. The players take turns to make a move, with Amy going first. Each move consists of choosing one of the piles available, removing the unchosen pile(s) from the game, and then dividing the chosen pile into 2 or 3 non-empty piles. A player loses the game if they are unable to make a move.

   Prove that Bec can always win the game, no matter how Amy plays.

3. Let ABC be a triangle with $\angle ACB = 90^\circ$. Suppose that the tangent line at C to the circle passing through A, B, C intersects the line AB at D. Let E be the midpoint of CD and let F be the point on the line EB such that AF is parallel to CD.

   Prove that the lines AB and CF are perpendicular.

4. Define the sequence $A_1, A_2, A_3, \ldots$ by $A_1 = 1$ and for $n = 1, 2, 3, \ldots$

$$A_{n+1} = \frac{A_n + 2}{A_n + 1}.$$

   Define the sequence $B_1, B_2, B_3, \ldots$ by $B_1 = 1$ and for $n = 1, 2, 3, \ldots$

$$B_{n+1} = \frac{B_n^2 + 2}{2B_n}.$$

   Prove that $B_{n+1} = A_{2^n}$ for all non-negative integers n.

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The Olympiad programs receive grant funding from the Australian Government.

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DAY 2
Wednesday, 5 February 2020
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. Each term of an infinite sequence $a_1, a_2, a_3, \ldots$ is equal to 0 or 1. For each positive integer n,

   (i) $a_n + a_{n+1} \neq a_{n+2} + a_{n+3}$, and

   (ii) $a_n + a_{n+1} + a_{n+2} \neq a_{n+3} + a_{n+4} + a_{n+5}$.

   Prove that if $a_1 = 0$, then $a_{2020} = 1$.

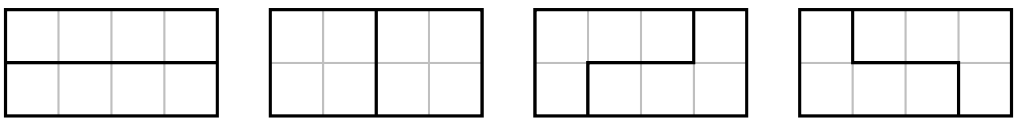
6. Let $ABCD$ be a square. For a point P inside $ABCD$, a *windmill* centred at P consists of two perpendicular lines ℓ_1 and ℓ_2 passing through P, such that

   - ℓ_1 intersects the sides AB and CD at W and Y, respectively, and
   - ℓ_2 intersects the sides BC and DA at X and Z, respectively.

   A windmill is called *round* if the quadrilateral $WXYZ$ is cyclic.

   Determine all points P inside $ABCD$ such that every windmill centred at P is round.

7. A *tetromino tile* is a tile that can be formed by gluing together four unit square tiles, edge to edge. For each positive integer n, consider a bathroom whose floor is in the shape of a $2 \times 2n$ rectangle. Let T_n be the number of ways to tile this bathroom floor with tetromino tiles. For example, $T_2 = 4$ since there are four ways to tile a 2×4 rectangular bathroom floor with tetromino tiles, as shown below.

   Prove that each of the numbers $T_1, T_2, T_3, \ldots$ is a perfect square.

8. Prove that for each integer k satisfying $2 \leq k \leq 100$, there are positive integers $b_2, b_3, \ldots, b_{101}$ such that

$$b_2^2 + b_3^3 + \cdots + b_k^k = b_{k+1}^{k+1} + b_{k+2}^{k+2} + \cdots + b_{101}^{101}.$$

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