AUSTRALIAN MATHS TRUST

2020 Australian Mathematical Olympiad

DAY 1
Tuesday, 4 February 2020
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Determine all pairs (a, b) of non-negative integers such that

$$\frac{a+b}{2} - \sqrt{ab} = 1.$$

2. Amy and Bec play the following game. Initially, there are three piles, each containing 2020 stones. The players take turns to make a move, with Amy going first. Each move consists of choosing one of the piles available, removing the unchosen pile(s) from the game, and then dividing the chosen pile into 2 or 3 non-empty piles. A player loses the game if they are unable to make a move.

   Prove that Bec can always win the game, no matter how Amy plays.

3. Let ABC be a triangle with $\angle ACB = 90°$. Suppose that the tangent line at C to the circle passing through A, B, C intersects the line AB at D. Let E be the midpoint of CD and let F be the point on the line EB such that AF is parallel to CD.

   Prove that the lines AB and CF are perpendicular.

4. Define the sequence $A_1, A_2, A_3, \ldots$ by $A_1 = 1$ and for $n = 1, 2, 3, \ldots$

$$A_{n+1} = \frac{A_n + 2}{A_n + 1}.$$

   Define the sequence $B_1, B_2, B_3, \ldots$ by $B_1 = 1$ and for $n = 1, 2, 3, \ldots$

$$B_{n+1} = \frac{B_n^2 + 2}{2B_n}.$$

   Prove that $B_{n+1} = A_{2^n}$ for all non-negative integers n.

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The Olympiad programs receive grant funding from the Australian Government.

AUSTRALIAN MATHS TRUST

2020 Australian Mathematical Olympiad

DAY 2
Wednesday, 5 February 2020
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. Each term of an infinite sequence $a_1, a_2, a_3, \ldots$ is equal to 0 or 1. For each positive integer n,

   (i) $a_n + a_{n+1} \neq a_{n+2} + a_{n+3}$, and

   (ii) $a_n + a_{n+1} + a_{n+2} \neq a_{n+3} + a_{n+4} + a_{n+5}$.

   Prove that if $a_1 = 0$, then $a_{2020} = 1$.

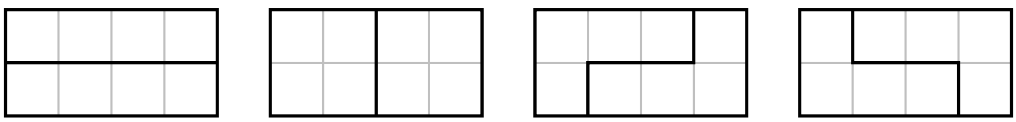
6. Let $ABCD$ be a square. For a point P inside $ABCD$, a *windmill* centred at P consists of two perpendicular lines ℓ_1 and ℓ_2 passing through P, such that

   - ℓ_1 intersects the sides AB and CD at W and Y, respectively, and
   - ℓ_2 intersects the sides BC and DA at X and Z, respectively.

   A windmill is called *round* if the quadrilateral $WXYZ$ is cyclic.

   Determine all points P inside $ABCD$ such that every windmill centred at P is round.

7. A *tetromino tile* is a tile that can be formed by gluing together four unit square tiles, edge to edge. For each positive integer n, consider a bathroom whose floor is in the shape of a $2 \times 2n$ rectangle. Let T_n be the number of ways to tile this bathroom floor with tetromino tiles. For example, $T_2 = 4$ since there are four ways to tile a 2×4 rectangular bathroom floor with tetromino tiles, as shown below.



   Prove that each of the numbers $T_1, T_2, T_3, \ldots$ is a perfect square.

8. Prove that for each integer k satisfying $2 \leq k \leq 100$, there are positive integers $b_2, b_3, \ldots, b_{101}$ such that

$$b_2^2 + b_3^3 + \cdots + b_k^k = b_{k+1}^{k+1} + b_{k+2}^{k+2} + \cdots + b_{101}^{101}.$$

The Olympiad programs receive grant funding from the Australian Government.

1. Determine all pairs (a, b) of non-negative integers such that

$$\frac{a+b}{2} - \sqrt{ab} = 1.$$

Solution 1 (Chris Wetherell)

Without loss of generality, we assume that $a \geq b$. Doubling both sides and factorising gives

$$\begin{aligned} a - 2\sqrt{ab} + b &= 2 \\ (\sqrt{a} - \sqrt{b})^2 &= 2 \\ \sqrt{a} - \sqrt{b} &= \sqrt{2}. \end{aligned}$$

We ignore the negative root since $a \geq b$.

Solving for a and rearranging gives

$$\begin{aligned} \sqrt{a} &= \sqrt{b} + \sqrt{2} && (\dagger) \\ a &= b + 2\sqrt{2b} + 2 \\ 2\sqrt{2b} &= a - b - 2, \end{aligned}$$

hence $2\sqrt{2b}$ must be a non-negative integer. In order for $2b$ to be a perfect square, b must be even. Moreover, $b/2$ must be a perfect square as well. Hence $b = 2n^2$, where $n \geq 0$.

Substituting into $(\dagger)$ gives

$$\sqrt{a} = \sqrt{2n^2} + \sqrt{2} = n\sqrt{2} + \sqrt{2} = (n+1)\sqrt{2},$$

hence $a = 2(n+1)^2$.

Therefore the pair $\{a, b\}$ must be of the form $\{2(n+1)^2, 2n^2\}$ for some integer $n \geq 0$. Finally, we verify that all such pairs do indeed satisfy the original condition:

$$\frac{2(n+1)^2 + 2n^2}{2} - \sqrt{2(n+1)^2 \times 2n^2} = (n+1)^2 + n^2 - 2n(n+1) = 1.$$

Solution 2 (Angelo Di Pasquale)

Without loss of generality, suppose $a \leq b$. We have $\frac{a+b}{2} = \sqrt{ab} + 1$. Hence $\sqrt{ab}$ is rational. Since it is rational and a, b are integers, it follows that $\sqrt{ab}$ is an integer.

Let $a = m^2 k$ where k is the square-free part of a. Then $b = n^2 k$ for some positive integer $n \geq m$. Substituting these into the original condition yields

$$\frac{m^2k + n^2k}{2} = mnk + 1 \qquad \Longleftrightarrow \qquad (n-m)^2 k = 2.$$

It follows that $n = m + 1$ and $k = 2$. So $(a, b) = (2m^2, 2(m+1)^2)$ which can be verified to satisfy the original condition.

Solution 3 (Kevin McAvaney)

Squaring both sides of $(a+b)/2-1=\sqrt{ab}$ then rearranging yields

$$a^2-(4+2b)a+(b-2)^2=0.$$

Solving the quadratic for a gives

$$a=(2+b)\pm\sqrt{8b}.$$

Since a is an integer, $b=2n^2$ for some non-negative integer n, and

$$a=2(n^2\pm 2c+1)=2(n\pm 1)^2.$$

Therefore all solutions are of the form $(2n^2,2(n+1)^2)$, which indeed satisfy the original condition.

2. Amy and Bec play the following game. Initially, there are three piles, each containing 2020 stones. The players take turns to make a move, with Amy going first. Each move consists of choosing one of the piles available, removing the unchosen pile(s) from the game, and then dividing the chosen pile into 2 or 3 non-empty piles. A player loses the game if they are unable to make a move.

   Prove that Bec can always win the game, no matter how Amy plays.

   Solution (Angelo Di Pasquale)

   Call a pile *perilous* if the number of stones in it is one more than a multiple of three, and *safe* otherwise. Bec has a winning strategy by ensuring that she only leaves Amy perilous piles. Bec wins because the number of stones is strictly decreasing, and eventually Amy will be left with two or three piles each with just one stone.

   To see that this is a winning strategy, we prove that Bec can always leave Amy with only perilous piles, and that under such circumstances, Amy must always leave Bec with at least one safe pile.

   On Amy's turn, whenever all piles are perilous it is impossible to choose one such perilous pile and divide it into two or three perilous piles by virtue of the fact that $1 + 1 \not\equiv 1 \pmod 3$ and $1 + 1 + 1 \not\equiv 1 \pmod 3$. Thus Amy must leave Bec with at least one safe pile.

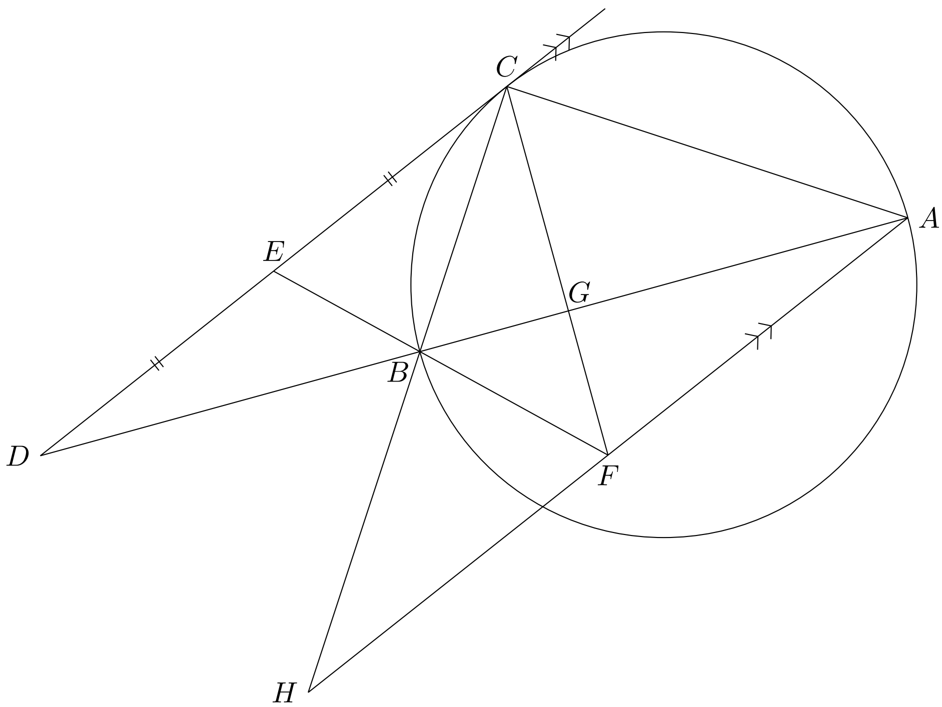
   On Bec's turn, whenever one of the piles is safe, she can divide it into two or three piles, each of which are safe, by virtue of the fact that $2 \equiv 1 + 1 \pmod 3$ and $0 \equiv 1 + 1 + 1 \pmod 3$.

3. Let ABC be a triangle with $\angle ACB = 90°$. Suppose that the tangent line at C to the circle passing through A, B, C intersects the line AB at D. Let E be the midpoint of CD and let F be the point on the line EB such that AF is parallel to CD.

   Prove that the lines AB and CF are perpendicular.

Solution 1 (Alan Offer)

Let BC and AF meet at H. Since CD and AH are parallel, triangles BCE and BHF are similar, and so too are triangles BDE and BAF. Hence, $AF/DE = BF/BE = FH/EC = FH/DE$, where the last equation holds because E is the midpoint of CD. It follows that $AF = FH$.



Since $\angle ACH$ is a right angle, the circumcircle of triangle ACH has AH as diameter and so F as centre. Hence, $FC = FH$ as these are both radii. It follows that triangle CFH is isosceles, so $\angle BCF = \angle BHF = \angle BCD = \angle BAC$, where the alternate segment theorem is used for both the second and fourth equalities.

Letting G be the point of intersection between AB and CF, it follows that triangles ABC and CBG are similar, as two pairs of corresponding angles are equal. Hence, $\angle BGC = \angle BCA$, which is a right angle, as required.

Solution 2 (Angelo Di Pasquale)

Let AB and CF intersect at G. Menelaus' theorem applied to $\triangle CDG$ with transversal line EBF yields

$$\frac{GB}{BD} \cdot \frac{DE}{EC} \cdot \frac{CF}{GF} = 1. \tag{1}$$

Note that

$$\frac{CF}{GF} = \frac{CG + GF}{GF} = \frac{CG}{GF} + 1 = \frac{DG}{GA} + 1 = \frac{DG + GA}{GA} = \frac{DA}{GA}.$$

Putting this into (1), using $DE = EC$, and rearranging yields

$$GB \cdot DA = GA \cdot BD. \tag{2}$$

Let O be the centre of circle ABC, and r its radius. Then (2) is equivalent to

$$\begin{aligned}(r-OG)(r+OD)=(r+OG)(OD-r) \quad &\iff \quad OG\cdot OD=r^2\\ &\iff \quad \frac{OG}{OC}=\frac{OC}{OD}.\end{aligned}$$

The last equality above implies $\triangle OGC \sim \triangle OCD$. Hence $\angle OGC = \angle OCD = 90°$.

Solution 3 (Alice Devillers)

The diameter of the circumcircle is AB. Without loss of generality, we can assume the radius is 1. Set $O=(0,0)$, $A=(-1,0)$, $B=(1,0)$ and $C=(c,s)$ where $c=\cos\alpha, s=\sin\alpha$ for some $\alpha\in(0,\pi)\setminus\{\pi/2\}$.

Note that the tangent at C has gradient $-\frac{c}{s}$. Then $D=(x,0)$ where

$$0-s=-\frac{c}{s}(x-c) \quad \Longrightarrow \quad x=c+\frac{s^2}{c}=\frac{1}{c}.$$

Hence the midpoint of CD is given by $E=(\frac{1}{2c}+\frac{c}{2},\frac{s}{2})=(\frac{1+c^2}{2c},\frac{s}{2})$.

To compute F, the line through A parallel to CD has equation $y=-\frac{c}{s}(x-1)$, while the line EB has equation

$$y=\frac{\frac{s}{2}-0}{\frac{1+c^2}{2c}+1}(x+1)=\frac{sc}{(1+c)^2}(x+1).$$

Solving them simultaneously,

$$\begin{aligned}\frac{sc}{(1+c)^2}(x+1)=-\frac{c}{s}(x-1) \quad &\Longrightarrow \quad s^2(x+1)=(1+c)^2(1-x)\\ &\Longrightarrow \quad x=\frac{(1+c)^2-s^2}{(1+c)^2+s^2}=\frac{2c+2c^2}{2+2c}=c.\end{aligned}$$

Hence the x-coordinate of F is c, which is the same as C. Therefore CF is perpendicular to AB as required.

Solution 4 (Angelo Di Pasquale)

There are also several solution approaches using projective techniques. Here we present one such variation.

Let G be the point where CF and AB meet, and let Q be the point 'at infinity' on the line CD. Since E is the midpoint of CD, $(C,D;E,Q)$ are harmonic conjugates. Projecting from F, it follows that $(G,D;B,A)$ are harmonic conjugates. Hence CF is the polar of D with respect to the circle and must be perpendicular to the diameter AB.

4. Define the sequence $A_1, A_2, A_3, \ldots$ by $A_1 = 1$ and for $n = 1, 2, 3, \ldots$

$$A_{n+1} = \frac{A_n + 2}{A_n + 1}.$$

Define the sequence $B_1, B_2, B_3, \ldots$ by $B_1 = 1$ and for $n = 1, 2, 3, \ldots$

$$B_{n+1} = \frac{B_n^2 + 2}{2B_n}.$$

Prove that $B_{n+1} = A_{2^n}$ for all non-negative integers n.

Solution 1 (Alice Devillers)

We will prove the following claim by induction. For all n,

$$A_{2n} = \frac{A_n^2 + 2}{2A_n}.$$

The base case holds for $n = 1$ as $A_1 = 1$ and $A_2 = \frac{3}{2}$. Assume the claim holds for n, we now show it holds for $n + 1$. This can be shown via the following computations. Let $A_n = x$, then

$$A_{n+1} = \frac{x+2}{x+1}, \quad A_{2n+1} = \frac{A_{2n} + 2}{A_{2n} + 1} = \frac{\frac{x^2+2}{2x} + 2}{\frac{x^2+2}{2x} + 1} = \frac{x^2 + 4x + 2}{x^2 + 2x + 2},$$

$$A_{2(n+1)} = A_{2n+2} = \frac{\frac{x^2+4x+2}{x^2+2x+2} + 2}{\frac{x^2+4x+2}{x^2+2x+2} + 1} = \frac{3x^2 + 8x + 6}{2x^2 + 6x + 4} = \frac{(x+2)^2 + 2(x+1)^2}{2(x+2)(x+1)}$$

$$= \frac{\left(\frac{x+2}{x+1}\right)^2 + 2}{2\left(\frac{x+2}{x+1}\right)} = \frac{A_{n+1}^2 + 2}{2A_{n+1}},$$

which proves the claim.

Now it is straightforward to prove $B_{n+1} = A_{2^n}$ by another induction. It is true for $n = 0$ as $B_{0+1} = 1 = A_{2^0}$. Assume true for n, we show that it is also true for $n + 1$ using the claim above:

$$B_{n+2} = \frac{B_{n+1}^2 + 2}{2B_{n+1}} = \frac{A_{2^n}^2 + 2}{2A_{2^n}} = A_{2 \times 2^n} = A_{2^{n+1}}.$$

This completes the solution.

Solution 2 (Chaitanya Rao)

After writing initial terms we derive a recurrence that defines the numerator and denominator of A_n and that leads us to the following conjecture. The solution for A_n is

$$A_n := \sqrt{2}\frac{\alpha^n + \beta^n}{\alpha^n - \beta^n}$$

where $\alpha = (1 + \sqrt{2})$ and $\beta = (1 - \sqrt{2})$.

We prove this by induction. For $n = 1$,

$$A_1 = \sqrt{2}\frac{\alpha^1 + \beta^1}{\alpha^1 - \beta^1} = \sqrt{2}\frac{2}{2\sqrt{2}} = 1.$$

Assume the formula holds for A_n, we then look at A_{n+1}

$$\begin{aligned}
A_{n+1} = \frac{A_n+2}{A_n+1} &= 1 + \frac{1}{A_n+1} \\
&= 1 + \frac{1}{\sqrt{2}\frac{\alpha^n+\beta^n}{\alpha^n-\beta^n}+1} \\
&= 1 + \frac{\alpha^n-\beta^n}{\sqrt{2}(\alpha^n+\beta^n)+(\alpha^n-\beta^n)} \\
&= \frac{\sqrt{2}(\alpha^n+\beta^n)+2(\alpha^n-\beta^n)}{\sqrt{2}(\alpha^n+\beta^n)+(\alpha^n-\beta^n)} \\
&= \frac{\sqrt{2}\left[(1+\sqrt{2})\alpha^n+(1-\sqrt{2})\beta^n)\right]}{(1+\sqrt{2})\alpha^n-(1-\sqrt{2})\beta^n} \\
&= \sqrt{2}\frac{\alpha^{n+1}+\beta^{n+1}}{\alpha^{n+1}-\beta^{n+1}},
\end{aligned}$$

and the induction is complete.

Next, we note that

$$\begin{aligned}
A_{2n} &= \sqrt{2}\frac{\alpha^{2n}+\beta^{2n}}{\alpha^{2n}-\beta^{2n}} \\
&= \sqrt{2}\frac{(\alpha^n+\beta^n)^2+(\alpha^n-\beta^n)^2}{2(\alpha^n+\beta^n)(\alpha^n-\beta^n)} \\
&= \sqrt{2}\frac{\alpha^n+\beta^n}{2(\alpha^n-\beta^n)} + \frac{\alpha^n-\beta^n}{\sqrt{2}(\alpha^n+\beta^n)} \\
&= \frac{A_n}{2} + \frac{1}{A_n} \\
&= \frac{A_n^2+2}{2A_n}.
\end{aligned}$$

It follows that $A_{2^n} = \frac{A_{2^{n-1}}^2+2}{2A_{2^{n-1}}}$. If we define $d_n = A_{2^{n-1}}$ for $n = 1, 2, \ldots$ we find that $d_1 = A_1 = 1$ and $d_{n+1} = \frac{d_n^2+2}{2d_n}$. This is the same recurrence that defines B_n and we conclude that $B_{n+1} = d_{n+1} = A_{2^n}$ for non-negative integers n.

Remark The problem statement still holds if we replace the original recurrences by

$$A_{n+1} = \frac{A_n+k}{A_n+1}, \quad B_{n+1} = \frac{B_n^2+k}{2B_n}.$$

5. Each term of an infinite sequence $a_1, a_2, a_3, \ldots$ is equal to 0 or 1. For each positive integer n,

   (i) $a_n + a_{n+1} \neq a_{n+2} + a_{n+3}$, and

   (ii) $a_n + a_{n+1} + a_{n+2} \neq a_{n+3} + a_{n+4} + a_{n+5}$.

   Prove that if $a_1 = 0$, then $a_{2020} = 1$.

Solution (Daniel Mathews)

First, there can never be three consecutive terms 0,1,0. For if there were, then the next term must be 0 (else $0{+}1 = 0{+}1$), then 0 (else $1{+}0 = 0{+}1$), then 1 (else $0{+}0 = 0{+}0$). But then we have six consecutive terms 0,1,0,0,0,1, a contradiction since $0+1+0 = 0+0+1$.

The same argument with 0s and 1s reversed shows there can never be any three consecutive terms 1,0,1.

Consider the sequence as consisting of a block of 0s, then a block of 1s, then a block of 0s, and so on. If a block (other than the first) has size 1, then we have 3 consecutive terms 1,0,1 or 0,1,0, a contradiction. If a block (other than the first) has size 2, then we have 4 consecutive terms 0,1,1,0 or 1,0,0,1, a contradiction since $0+1 = 1+0$ and $1+0 = 0+1$. And if a block has size 4 or more, then we have 4 consecutive terms 0,0,0,0 or 1,1,1,1, a contradiction since $0+0 = 0+0$ and $1+1 = 1+1$.

Thus the first block has length at most 3, and every subsequent block has length exactly 3. Accordingly, for all i, a_{i+3} is different from a_i. Since $2020-1 = 2019$ is an odd multiple of 3, a_1 is different from a_{2020}. So $a_1 = 0$ implies $a_{2020} = 1$.

6. Let $ABCD$ be a square. For a point P inside $ABCD$, a *windmill* centred at P consists of two perpendicular lines ℓ_1 and ℓ_2 passing through P, such that

- ℓ_1 intersects the sides AB and CD at W and Y, respectively, and
- ℓ_2 intersects the sides BC and DA at X and Z, respectively.

A windmill is called *round* if the quadrilateral $WXYZ$ is cyclic.

Determine all points P inside $ABCD$ such that every windmill centred at P is round.

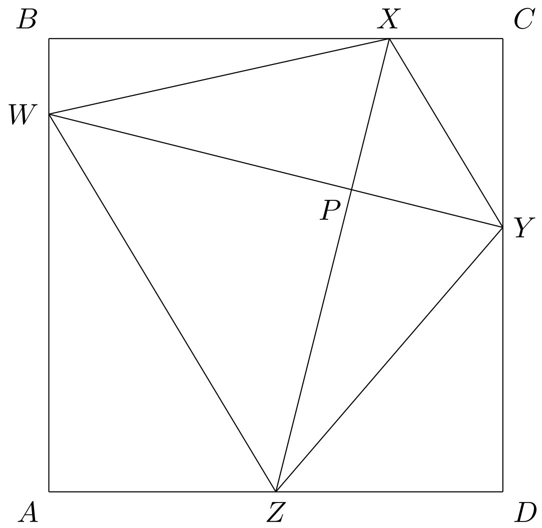
Solution 1 (Norman Do)

Since $\angle WBX = \angle WPX = 90°$, we know that the quadrilateral $WBXP$ is cyclic. Similarly, since $\angle YDZ = \angle YPZ = 90°$, we know that the quadrilateral $YDZP$ is cyclic.

Suppose that the quadrilateral $WXYZ$ is cyclic. Then we have the following equal angles.

$$\angle ABP = \angle WBP = \angle WXP = \angle WXZ = \angle ZYW = \angle ZYP = \angle ZDP = \angle ADP$$

Therefore, triangles ABP and ADP share the common side AP, have the equal sides $AB = AD$, and have the equal angles $\angle ABP = \angle ADP$. It follows that either $\angle APB = \angle APD$ or $\angle APB + \angle APD = 180°$. In the first case, triangles ABP and ADP are congruent, so P must lie on the segment AC. In the second case, P must lie on the segment BD. Therefore, P lies on one of the diagonals of the square $ABCD$.



Conversely, suppose that P lies on one of the diagonals of the square $ABCD$. In fact, we may assume without loss of generality that P lies on AC. Then the triangles ABP and ADP are congruent and we have the following equal angles.

$$\angle WXZ = \angle WXP = \angle WBP = \angle ABP = \angle ADP = \angle ZDP = \angle ZYP = \angle ZYW$$

Since $\angle WXZ = \angle ZYW$, it follows that the quadrilateral $WXYZ$ is cyclic.

Solution 2 (Angelo Di Pasquale)

The following is an alternative to the "hard part" of the problem only, that is, proving that P lies on the diagonal if $WXYZ$ is cyclic.

If P does not lie on the diagonal BD, then without loss of generality, P lies inside triangle ABD. If $WXYZ$ is cyclic, then we deduce as in the official solution that $\angle ABP =$

$\angle ADP = \alpha$, say. Thus we also have $\angle PBD = \angle PDB = 45° - \alpha$. Therefore, triangle PDB is isosceles with $PD = PB$. Hence P lies on the perpendicular bisector of BD, that is, P lies on AC. Thus if $WXYZ$ is cyclic, then P must lie on one of the diagonals.

Solution 3 (Kevin McAvaney)

Suppose P has coordinates (a, b) and lies inside the square $ABCD$, where $A = (0, 0)$, $B = (0, 1)$, $C = (1, 1)$, and $D = (1, 0)$. Let θ be the angle between ℓ_1 and AB. Since $\ell_1 \perp \ell_2$, θ must also be the angle between ℓ_2 and BC. Then

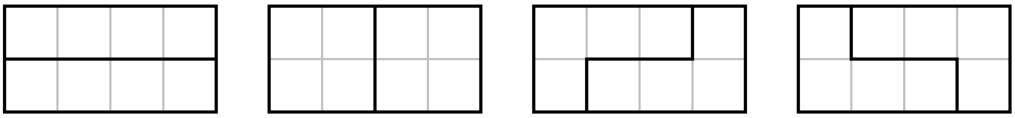
$$PW = \frac{a}{\sin\theta}, \quad PY = \frac{1-a}{\sin\theta}, \quad PZ = \frac{b}{\sin\theta}, \quad PX = \frac{1-b}{\sin\theta}.$$

Then, by power of a point, $WXYZ$ if and only if $PW \times PY = PZ \times PX$, or equivalently,

$$a(1-a) = b(1-b) \quad \iff \quad (a+b-1)(a-b) = 0.$$

This occurs when $a = b$ or $a + b = 1$, which correspond to the diagonals of $ABCD$.

7. A *tetromino tile* is a tile that can be formed by gluing together four unit square tiles, edge to edge. For each positive integer n, consider a bathroom whose floor is in the shape of a $2 \times 2n$ rectangle. Let T_n be the number of ways to tile this bathroom floor with tetromino tiles. For example, $T_2 = 4$ since there are four ways to tile a 2×4 rectangular bathroom floor with tetromino tiles, as shown below.



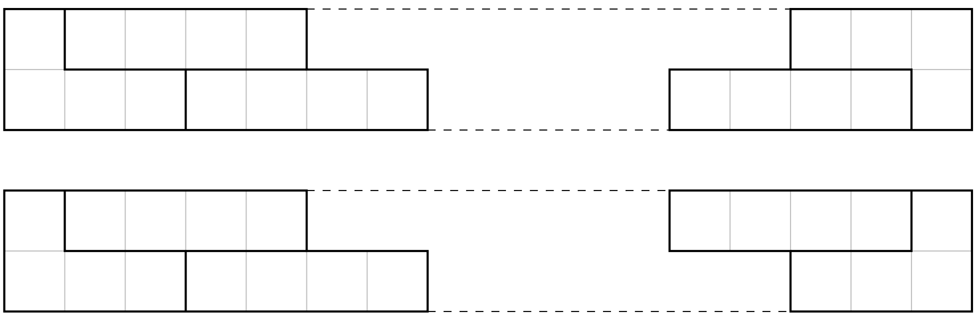
Prove that each of the numbers $T_1, T_2, T_3, \ldots$ is a perfect square.

Solution 1 (Sean Gardiner)

Let F_n denote the Fibonacci sequence, defined by $F_0 = 1$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. We will prove that $T_n = F_n^2$.

Consider a tiling of a $2 \times 2n$ rectangle with tetrominoes, where $n \geq 2$. Consider the behaviour of the tiling in the leftmost column of the rectangle. Exactly one of the following three cases must arise.

- The leftmost column is covered by a 2×2 square, whose removal leaves one of the T_{n-1} tilings of the $2 \times 2(n-1)$ rectangle.
- The leftmost column is covered by two 1×4 rectangles, whose removal leaves one of the T_{n-2} tilings of the $2 \times 2(n-2)$ rectangle.
- The leftmost column is covered by an L-tetromino and resembles one of the following diagrams, or their reflections in a horizontal axis. The areas outlined by dashed lines are tiled with 1×4 rectangles. (Observe that there may actually be any non-negative integer number of 1×4 rectangles appearing in the diagram.) The top case occurs when the area covers a number of columns that is 0 modulo 4, while the bottom case occurs when the area covers a number of columns that is 2 modulo 4. The removal of these areas leaves one of the T_{n-k} tilings of the $2 \times 2(n-k)$ rectangle, where $k = 2, 3, \ldots, n$. The case $k = n$ leaves zero columns, so we set $T_0 = 1$ to allow for this case.



Hence, we have shown that the following recursion holds for $n \geq 2$:

$$T_n = T_{n-1} + T_{n-2} + 2(T_0 + T_1 + \cdots + T_{n-2}).$$

Using this equation, one can directly verify that for $n \geq 3$,

$$T_n - 2T_{n-1} - 2T_{n-2} + T_{n-3} = 0.$$

Now observe that

$$F_n^2 + F_{n-3}^2 = (F_{n-1} + F_{n-2})^2 + (F_{n-1} - F_{n-2})^2 = 2F_{n-1}^2 + 2F_{n-2}^2.$$

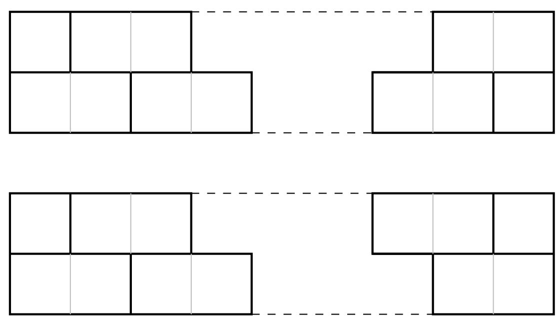
We can check that $T_n = F_n^2$ for small values of n and then use these two matching recursions to deduce that $T_n = F_n^2$ for all positive integers n.

Solution 2 (Norman Do)

We use the known fact that a tiling of a $1 \times n$ rectangle with unit squares and dominoes is enumerated by the Fibonacci number F_n, as defined in the previous solution. Therefore, the number of tilings of a $2 \times n$ rectangle with unit squares and horizontal dominoes is simply F_n^2. We will provide a bijection between the tilings of a $2 \times n$ rectangle with unit squares and horizontal dominoes and the tilings of a $2 \times 2n$ rectangle with tetrominoes.

Consider a tiling of a $2 \times n$ rectangle with unit squares and horizontal dominoes. Divide it into pieces by cutting along all vertical lines that do not pass through a domino.

- If a piece consists of two unit squares on top of each other, then we replace it with a 2×2 rectangle.
- If a piece consists of two dominoes on top of each other, then we replace it with two 1×4 rectangles on top of each other.
- Otherwise, the piece must resemble one of the following diagrams, or their reflections in a horizontal axis. The areas outlined by dashed lines are tiled with dominoes. (Observe that there may actually be any positive integer number of dominoes appearing in the diagram.) In this case, we replace the diagram with the respective diagrams from the previous solution.



The construction gives the necessary bijection between the tilings of a $2 \times n$ rectangle with unit squares and horizontal dominoes and the tilings of a $2 \times 2n$ rectangle with tetrominoes. Therefore, we conclude that $T_n = F_n^2$.

8. Prove that for each integer k satisfying $2 \leq k \leq 100$, there are positive integers $b_2, b_3, \ldots, b_{101}$ such that

$$b_2^2 + b_3^3 + \cdots + b_k^k = b_{k+1}^{k+1} + b_{k+2}^{k+2} + \cdots + b_{101}^{101}.$$

Solution (Ivan Guo)

Consider the equation

$$a_2^2 + \cdots + a_k^k - a_{k+1}^{k+1} - \cdots - a_{100}^{100} = L.$$

First of all, choose $a_2, \ldots, a_{100}$ arbitrarily so that L is positive (e.g., this can be achieved by making a_2 very big). Since 101 is coprime to 100!, there exist positive integers c and d such that $100!c + 1 = 101d$ (e.g., by setting c to be the inverse of $-100!$ in modulo 101). In fact, by Wilson's theorem, $100! + 1$ is divisible by 101, so $c = 1$ works.

Multiplying both sides by $L^{100!c}$, we have

$$(a_2 L^{\frac{100!c}{2}})^2 + \cdots + (a_k L^{\frac{100!c}{k}})^k - (a_{k+1} L^{\frac{100!c}{k+1}})^{k+1} - \cdots - (a_{100} L^{\frac{100!c}{100}})^{100} = (L^d)^{101}.$$

Therefore, setting $b_i = a_i L^{\frac{100!c}{i}}$ for $2 \leq i \leq 100$ and $b_{101} = L^d$ would satisfy the required condition.

Score Distribution/Problem

Number of Students/Score	Problem Number							
	1	2	3	4	5	6	7	8
0	2	57	101	151	10	86	98	155
1	18	3	24	4	6	10	30	16
2	29	3	4	6	3	13	8	2
3	18	5	1	0	3	11	2	2
4	16	6	0	1	15	15	3	1
5	19	4	1	1	19	1	3	1
6	11	7	1	5	27	8	1	1
7	72	100	53	17	102	41	40	7
Average	**4.6**	**4.4**	**2.3**	**0.9**	**5.7**	**2.5**	**2.0**	**0.5**