2021 Australian Mathematical Olympiad

AUSTRALIAN MATHS TRUST

DAY 1
Wednesday, 3 February 2021
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Let a, b and c be positive integers. Vaughan arranges abc identical white unit cubes into an $a \times b \times c$ rectangular prism and paints the outside of the prism red. After disassembling the prism back into unit cubes, he notices that the number of faces of the unit cubes that are red is the same as the number that are white.

   Find all values that the product abc could take.

2. Let $ABCDE$ be a convex pentagon such that AC is perpendicular to BD and AD is perpendicular to CE.

   Prove that $\angle BAC = \angle DAE$ if and only if triangles ABC and ADE have equal areas.

3. Each square in a 2021×2021 grid of unit squares can be coloured either red or blue. We can adjust the colours of the squares with a sequence of moves. In each move, we choose a rectangle composed of unit squares, and change all of its red squares to blue and all of its blue squares to red.

   A *monochrome path* in the grid is a sequence of distinct unit squares of the same colour, such that each shares an edge with the next. A colouring of the grid is called *tree-like* if, for any two unit squares S and T of the same colour, there is a unique monochrome path whose first square is S and last square is T.

   Determine the minimum number of moves required to reach a tree-like colouring when starting from a colouring in which all unit squares are red.

4. Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients such that the leading coefficient of $P(x)$ is 1. Suppose that $P(n)^n$ divides $Q(n)^{n+1}$ for infinitely many positive integers n.

   Prove that $P(n)$ divides $Q(n)$ for infinitely many positive integers n.

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The Olympiad program is supported by the Australian Government Department of Industry, Science, Energy and Resources through the Science Competitions: Mathematics and Informatics Olympiads grant opportunity.

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DAY 2
Thursday, 4 February 2021
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. Determine all pairs (a, b) of real numbers that simultaneously satisfy the equations

$$a^{20} + b^{20} = 1 \qquad \text{and} \qquad a^{21} + b^{21} = 1.$$

6. A school has 60 students in year 2 who will be divided into three classes of 20 students. Each student writes a list of three other students that they hope to have in their class.

   Can the school always arrange for each student to be in the same class as at least one of the three students on their list?

7. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 1$ and for $n = 1, 2, 3, \ldots$

$$a_{n+1} = a_n^2 + 1.$$

   Prove that there exists a positive integer n such that a_n has a prime factor with more than 2021 digits.

8. Let ABC be a triangle with incentre I. Suppose that D is a variable point on the circumcircle of ABC, on the arc AB that does not contain C. Let E be a point on the line segment BC such that $\angle ADI = \angle IEC$.

   Prove that, as D varies, the line DE passes through a fixed point.

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