2021 Australian Mathematical Olympiad

AUSTRALIAN MATHS TRUST

DAY 1
Wednesday, 3 February 2021
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Let a, b and c be positive integers. Vaughan arranges abc identical white unit cubes into an $a \times b \times c$ rectangular prism and paints the outside of the prism red. After disassembling the prism back into unit cubes, he notices that the number of faces of the unit cubes that are red is the same as the number that are white.

   Find all values that the product abc could take.

2. Let $ABCDE$ be a convex pentagon such that AC is perpendicular to BD and AD is perpendicular to CE.

   Prove that $\angle BAC = \angle DAE$ if and only if triangles ABC and ADE have equal areas.

3. Each square in a 2021×2021 grid of unit squares can be coloured either red or blue. We can adjust the colours of the squares with a sequence of moves. In each move, we choose a rectangle composed of unit squares, and change all of its red squares to blue and all of its blue squares to red.

   A *monochrome path* in the grid is a sequence of distinct unit squares of the same colour, such that each shares an edge with the next. A colouring of the grid is called *tree-like* if, for any two unit squares S and T of the same colour, there is a unique monochrome path whose first square is S and last square is T.

   Determine the minimum number of moves required to reach a tree-like colouring when starting from a colouring in which all unit squares are red.

4. Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients such that the leading coefficient of $P(x)$ is 1. Suppose that $P(n)^n$ divides $Q(n)^{n+1}$ for infinitely many positive integers n.

   Prove that $P(n)$ divides $Q(n)$ for infinitely many positive integers n.

2021 Australian Mathematical Olympiad

AUSTRALIAN MATHS TRUST

DAY 2
Thursday, 4 February 2021
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. Determine all pairs (a, b) of real numbers that simultaneously satisfy the equations

$$a^{20} + b^{20} = 1 \qquad \text{and} \qquad a^{21} + b^{21} = 1.$$

6. A school has 60 students in year 2 who will be divided into three classes of 20 students. Each student writes a list of three other students that they hope to have in their class.

   Can the school always arrange for each student to be in the same class as at least one of the three students on their list?

7. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 1$ and for $n = 1, 2, 3, \ldots$

$$a_{n+1} = a_n^2 + 1.$$

   Prove that there exists a positive integer n such that a_n has a prime factor with more than 2021 digits.

8. Let ABC be a triangle with incentre I. Suppose that D is a variable point on the circumcircle of ABC, on the arc AB that does not contain C. Let E be a point on the line segment BC such that $\angle ADI = \angle IEC$.

   Prove that, as D varies, the line DE passes through a fixed point.

1. Let a, b and c be positive integers. Vaughan arranges abc identical white unit cubes into an $a \times b \times c$ rectangular prism and paints the outside of the prism red. After disassembling the prism back into unit cubes, he notices that the number of faces of the unit cubes that are red is the same as the number that are white.

   Find all values that the product abc could take.

Solution 1 (Mike Clapper)

Answer: $abc = 8, 16$ or 18.

The total number of faces of the small cubes is $6abc$ and the total number of painted faces is $2ab + 2bc + 2ca$. So we have

$$2ab + 2bc + 2ca = 3abc.$$

Assume, without loss of generality, that $a \le b \le c$.

If $a = 1$, then we have $2b + 2c + 2bc = 3bc$, so that $bc - 2b - 2c = 0$. This can be rearranged as $(b-2)(c-2) = 4$. Since b and c are positive integers, we have two solutions: $b-2 = 1, c-2 = 4$ and $b-2 = 2, c-2 = 2$ giving $(a, b, c) = (1, 3, 6)$ or $(1, 4, 4)$. Hence $abc = 18$ or 16.

If $a = 2$, then we have $4b + 4c + 2bc = 6bc$ so that $bc - b - c = 0$. Rearranging, we have $(b-1)(c-1) = 1$ which has only one solution in positive integers, $b = 2, c = 2$. This gives $abc = 8$.

If $a \ge 3$, then we have $3abc \ge 9bc > 2ab + 2bc + 2ca$. Hence there are no further solutions.

Solution 2 (Ivan Guo)

Again let us reduce the problem to finding positive integer solutions to the equation

$$2ab + 2bc + 2ca = 3abc.$$

The equation can be rewritten as

$$1/a + 1/b + 1/c = 3/2.$$

Assume, without loss of generality, that $a \le b \le c$. So $1/c \le 1/b \le 1/a$.

Then

$$3/2 = 1/a + 1/b + 1/c \le 3 \times 1/a \quad \Longleftrightarrow \quad a \le 2.$$

If $a = 2$, then we have equality above, which can only be achieved if $a = b = c = 2$. This yields the solution $abc = 8$.

If $a = 1$, then

$$1/b < 1/2 = 1/b + 1/c \le 2 \times 1/b \quad \Longleftrightarrow \quad 2 < b \le 4.$$

The only possibilities are $b = 3$ and $b = 4$, which lead to $c = 6$ and $c = 4$, respectively. So we also have the solutions $abc = 18$ and $abc = 16$.

Solution 3 (Angelo Di Pasquale)

Here is a geometrical argument. If at least two of the dimensions of the prism are 1, then every unit cube has more than three painted faces. So this cannot occur.

If exactly one of the dimensions of the prism is 1, then each of the four corner cubes have four painted faces, each edge cube has three painted faces, and each interior cube as two painted faces. This implies that there are exactly four interior cubes. So the four interior cubes form a prism of dimensions $1\times1\times4$ or $1\times2\times2$. So the whole prism has dimensions $1 \times 3 \times 6$ or $1 \times 4 \times 4$ which yield $abc = 18$ or 16.

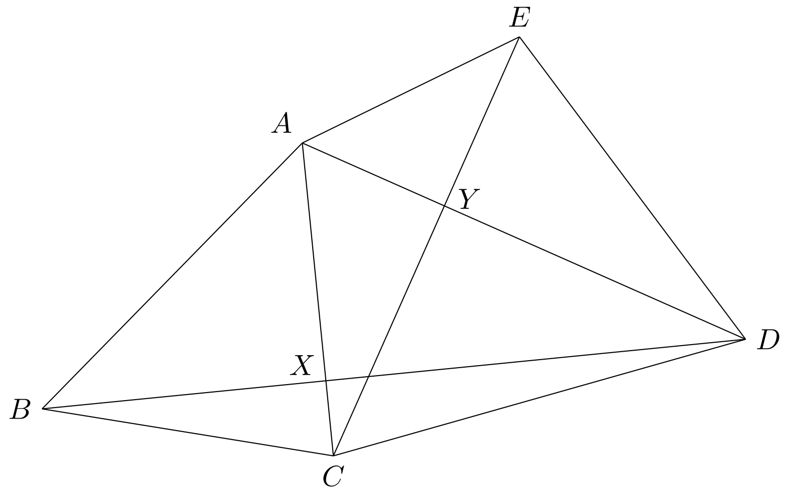
If all of the dimensions of the prism are at least 2, then each unit cube has at most three painted faces, with equality if and only if each unit cube is a corner cube. This yields $abc = 2 \times 2 \times 2 = 8$.

2. Let $ABCDE$ be a convex pentagon such that AC is perpendicular to BD and AD is perpendicular to CE.

   Prove that $\angle BAC = \angle DAE$ if and only if triangles ABC and ADE have equal areas.

Solution 1 (Ivan Guo)

Let X be the intersection of AC and BD and Y be the intersection of AD and CE.



The right angles from the problem imply that $XYCD$ is cyclic. By power of a point, $AX \times AC = AY \times AD$. Hence we have the following equivalences:

$$\begin{aligned} & \angle BAC = \angle DAE \\ \iff \quad & \triangle BAX \sim \triangle EAY \\ \iff \quad & BX/AX = EY/AY \\ \iff \quad & BX \times AC = EY \times AD \\ \iff \quad & |\triangle ABC| = |\triangle ADE|. \end{aligned}$$

Remark There are many variations of this solution, for example, by using similar triangles instead of the cyclic quadrilateral.

Solution 2 (Chaitanya Rao)

Define X and Y as in the first solution. Then $XYCD$ is cyclic, so $\angle ACE = \angle ADB$. Equipped with this, it follows that

$$\begin{aligned} & |\triangle ABC| = |\triangle ADE| \\ \iff \quad & |ABCD| = |ACDE| \\ \iff \quad & AC \times BD = AD \times CE \\ \iff \quad & AC/CE = AD/BD \\ \iff \quad & \triangle ACE \sim \triangle ADB \\ \iff \quad & \angle BAC = \angle DAE. \end{aligned}$$

Solution 3 (Ivan Guo)

Define X and Y as in the first solution. Once again, begin by noting that $XYCD$ is cyclic. So $\angle ACE = \angle YDB$.

Suppose that $\angle BAC = \angle DAE$. Then triangle ABD and AEC are similar. Then

$$\begin{aligned} AB/AD = AE/AC &\implies AB \times AC \sin(\angle BAC) = AD \times AE \sin(\angle DAE) \\ &\implies |\triangle ABC| = |\triangle ADE|. \end{aligned}$$

For the converse, we will use reverse reconstruction. Suppose triangles ABC and ADE have equal areas. Construct the point B' on BD such that $\angle B'AC = \angle DAE$ and $AB'CDE$ is convex. Then by the earlier arguments

$$|\triangle AB'C| = |\triangle ADE| \implies |\triangle AB'C| = |\triangle ABC| \implies B = B',$$

as required.

Solution 4 (Andrew Elvey Price)

Define X and Y as in the first solution. By applying Pythagoras' theorem to quadrilaterals with perpendicular diagonals,

$$AB^2 - BC^2 + AC^2 = AD^2 - DC^2 + AC^2 = AD^2 - ED^2 + AE^2.$$

Hence, the cosine rule implies that

$$BA \times AC \cos(\angle BAC) = DA \times AE \cos(\angle DAE).$$

Finally, since $\angle BAC$ and $\angle DAE$ are acute angles,

$$\begin{aligned} |\triangle ABC| = |\triangle ADE| &\iff BA \times AC \sin(\angle BAC) = DA \times AE \sin(\angle DAE) \\ &\iff \tan(\angle BAC) = \tan(\angle DAE) \iff \angle BAC = \angle DAE, \end{aligned}$$

as required.

Solution 5 (Alan Offer)

Let us use lowercase for complex numbers representing the points with the corresponding uppercase letter. Placing the origin at A, we then have $a = 0$.

In terms of complex numbers, the condition that $AC \perp BD$ is equivalent to $b\bar{c} + \bar{b}c = c\bar{d} + \bar{c}d$. Furthermore, this common value is nonzero since A, B and D are not collinear by the convexity of the pentagon. Similarly, from $AD \perp CE$ we have $c\bar{d} + \bar{c}d = d\bar{e} + \bar{d}e$. Thus we have

$$b\bar{c} + \bar{b}c = d\bar{e} + \bar{d}e \neq 0. \tag{1}$$

Next, the triangles ABC and ADE having equal areas is equivalent to $\bar{b}c$ and $\bar{d}e$ having equal imaginary parts, and so to

$$\bar{b}c - b\bar{c} = \bar{d}e - d\bar{e}. \tag{2}$$

Finally, the angles $\angle BAC$ and $\angle DAE$ being equal is equivalent to $(b/c)/(d/e) = (be)/(cd)$ being real, and so

$$b\bar{c}\bar{d}e = \bar{b}cd\bar{e}. \tag{3}$$

Thus, we need to show that, given (1), the statements (2) and (3) are equivalent. To this end, we have

$$\begin{aligned}
& & \bar{b}c - b\bar{c} &= \bar{d}e - d\bar{e} \\
&\iff & (\bar{b}c - b\bar{c})(\bar{d}e + d\bar{e}) &= (\bar{d}e - d\bar{e})(\bar{b}c + b\bar{c}) \\
&\iff & \bar{b}c\bar{d}e + \bar{b}cd\bar{e} - b\bar{c}\bar{d}e - b\bar{c}d\bar{e} &= \bar{b}c\bar{d}e - \bar{b}cd\bar{e} + b\bar{c}\bar{d}e - b\bar{c}d\bar{e} \\
&\iff & 2\bar{b}cd\bar{e} &= 2b\bar{c}\bar{d}e \\
&\iff & \bar{b}cd\bar{e} &= b\bar{c}\bar{d}e.
\end{aligned}$$

3. Each square in a 2021×2021 grid of unit squares can be coloured either red or blue. We can adjust the colours of the squares with a sequence of moves. In each move, we choose a rectangle composed of unit squares, and change all of its red squares to blue and all of its blue squares to red.

A *monochrome path* in the grid is a sequence of distinct unit squares of the same colour, such that each shares an edge with the next. A colouring of the grid is called *tree-like* if, for any two unit squares S and T of the same colour, there is a unique monochrome path whose first square is S and last square is T.

Determine the minimum number of moves required to reach a tree-like colouring when starting from a colouring in which all unit squares are red.

Solution 1 (Ivan Guo)

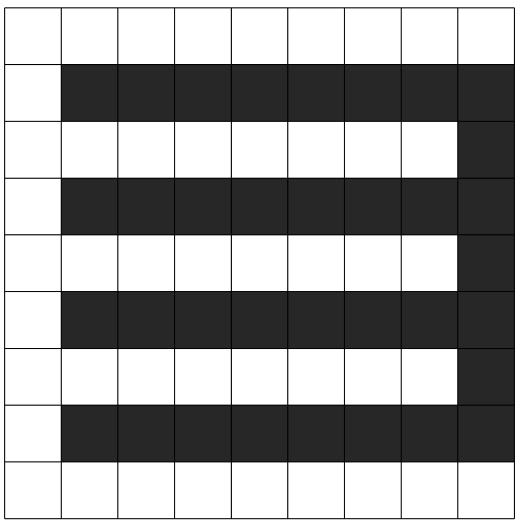
The answer for an $n \times n$ chessboard is $\lfloor n/2 \rfloor$. So for $n = 2021$, the answer is 1010.

We first show that at least $\lfloor n/2 \rfloor$ moves are needed. Suppose that we achieve the required state after $k \leq \lfloor n/2 \rfloor - 1$ moves. In the chessboard there are $n-1$ interior horizontal lines and $n-1$ interior vertical lines (excluding the perimeter of the chessboard). In each move, the perimeter of the chosen rectangle is made up of two vertical and two horizontal lines. Since $2k < n - 1$, after k moves, at least one vertical line, say v, and one horizontal line, say h, of the chessboard do not coincide with the perimeters of the k chosen rectangles. Hence the four unit squares adjacent to the intersection point of v and h have the same colour at the end of k moves. This is a contradiction since a monochrome 2×2 square cannot be part of a tree-like colouring.

It remains to show that $\lfloor n/2 \rfloor$ moves is sufficient. Suppose the chessboard is on the Cartesian plane, described by the region $0 \leq x, y \leq n$. The required state can be achieved by the following moves.

- For the first move, choose the rectangle defined by $1 \leq x \leq n$ and $1 \leq y \leq 2\lfloor n/2 \rfloor$.
- For the ith move where $i = 2, 3, \ldots, \lfloor n/2 \rfloor$, choose the rectangle defined by $1 \leq x \leq n - 1$ and $2i - 2 \leq y \leq 2i - 1$.

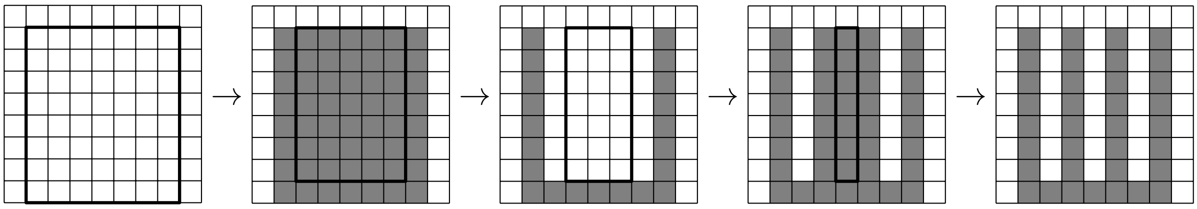
The following diagram shows the final configuration for the $n = 9$ case.



It is easy to check that, after $\lfloor n/2 \rfloor$ moves, the grid indeed has a tree-like colouring.

Solution 2 (Mike Clapper)

An alternative to the final construction. We illustrate how to do it for a 9×9 board with 4 moves. The principle is the same for any board of odd side length $2n+1$ using n moves.



After the second chosen rectangle, each subsequent rectangle is chosen by simply moving its vertical sides inwards by one unit each.

4. Let $P(x)$ and $Q(x)$ be polynomials with integer coefficients such that the leading coefficient of $P(x)$ is 1. Suppose that $P(n)^n$ divides $Q(n)^{n+1}$ for infinitely many positive integers n.

   Prove that $P(n)$ divides $Q(n)$ for infinitely many positive integers n.

Solution 1 (Michelle Chen)

Suppose that $P(n) \mid Q(n)$ only holds for finitely many positive integers n. Then, there are infinitely many n such that $P(n) \nmid Q(n)$ and $P(n)^n \mid Q(n)^{n+1}$. For each such n, $P(n) \nmid Q(n)$ implies that there exists some prime q and integer α such that $q^\alpha \mid Q(n)$ but $q^{\alpha+1} \nmid Q(n)$ and $q^{\alpha+1} \mid P(n)$. We then have $q^{\alpha(n+1)+1} \nmid Q(n)^{n+1}$ and $q^{\alpha n+n} \mid P(n)^n \mid Q(n)^{n+1}$, so that $\alpha n + n \le \alpha n + \alpha$, which is equivalent to $n \le \alpha$.

But then $q^n \mid q^\alpha \mid P(n)$, so $P(n) = 0$ or $2^n \le q^n \le |P(n)|$. As P is not the zero polynomial, this implies that $2^n \le |P(n)|$ for infinitely many n, which is a contradiction.

Solution 2 (Ian Wanless)

Let $R(n) = Q(n)/P(n)$. Suppose that $R(n) \notin \mathbb{Z}$ but $(R(n))^n Q(n) \in \mathbb{Z}$. Then $R(n) \in \mathbb{Q}\backslash\mathbb{Z}$ so $R(n) = q/p$ for relatively prime integers q and p, with $p \ge 2$. But then $Q(n)q^n/p^n \in \mathbb{Z}$, which implies that p^n divides $Q(n)$, and in particular $Q(n) \ge p^n \ge 2^n$. But Q is a polynomial so there are at most finitely many n for which $Q(n) \ge 2^n$. The result follows.

Solution 3 (Ivan Guo)

The statement is trivially true if $P(x) \equiv 1$. So we shall henceforth assume that $\deg(P) \ge 1$. It suffices to prove that $Q(x) \equiv A(x)P(x)$ for some integer polynomial $A(x)$.

First, for any positive integer k, if $n \ge k$ then

$$\begin{aligned} P(n)^n \mid Q(n)^{n+1} &\iff P(n)^{n(k+1)} \mid Q(n)^{(n+1)(k+1)} \\ &\implies P(n)^{(n+1)k} \mid Q(n)^{(n+1)(k+1)} \\ &\iff P(n)^k \mid Q(n)^{k+1}. \end{aligned}$$

Hence for any k, $P(n)^k \mid Q(n)^{k+1}$ holds for infinitely many positive n.

Next, we claim the above implies that $P(n)^k$ is a polynomial factor of $Q(n)^{k+1}$. Since P is monic, we can carry out polynomial division to obtain

$$Q(x)^{k+1} \equiv B(x)P(x)^k + R(x)$$

where $B(x), R(x)$ are integer polynomials with $\deg(R) < k\deg(P)$. Then, for infinitely many positive n,

$$P(n)^k \mid Q(n)^{k+1} \implies P(n)^k \mid R(n).$$

However, since $\deg(R) < k\deg(P)$, for all large enough x, $P(x)^k > |R(x)|$. This implies that $R(n) = 0$ for infinitely many n, hence $R(x) \equiv 0$. Therefore for any positive integer k, $P(n)^k$ is a polynomial factor of $Q(n)^{k+1}$.

Now suppose that $P(x)$ is not a polynomial factor of $Q(x)$. Then there exists a (possibly complex) zero of $P(x)$, say a, with multiplicity $d > 0$, such that $(x-a)^d$ is a factor of $P(x)$ but not a factor of $Q(x)$. Let the multiplicity of $(x-a)$ in $Q(x)$ be e, where $d > e \ge 0$. Then, since $P(n)^k$ is a factor of $Q(n)^{k+1}$, we must have, for all positive integers k,

$$dk \le e(k+1) \iff (d-e) \le e/k.$$

This is a contradiction since e/k can be made arbitrarily small. Therefore we must have $Q(x) \equiv A(x)P(x)$. Finally, by polynomial division, $A(x)$ is an integer polynomial because $P(x), Q(x)$ are integer polynomials and $P(x)$ is monic, completing the proof.

Solution 4 (Angelo Di Pasquale)

For each positive integer r, let

$$S_r = \{n \in \mathbb{N}^+ : P(n)^r \mid Q(n)^{r+1}\}.$$

As in the previous solution we deduce that S_r is infinite for each positive integer r.

Observe that if $n \in S_{r+1}$ we have

$$P(n)^{r+1} \mid Q(n)^{r+2} \quad \Rightarrow \quad P(n)^{r(r+1)} \mid Q(n)^{r(r+2)} \mid Q(n)^{(r+1)^2} \quad \Rightarrow \quad P(n)^r \mid Q(n)^{r+1}.$$

Hence $S_{r+1} \subseteq S_r$ for each r. So whenever n "works" for r, it also works for everything less than r.

By polynomial division, since P is monic, we may write $Q(x) = A(x)P(x) + R(x)$, where $\deg(R) < \deg(P) = t$.

We shall now prove by induction on $r \geq 0$ that

$$P(n)^r \mid R(n)^{r+1} \quad \text{for all } n \in S_r.$$

It is obviously true for $r = 0$.

Assume inductively that $P(n)^{r-1} \mid R(n)^r$ for all $n \in S_{r-1}$. For each $n \in S_r$, we have

$$P(n)^r \mid Q(n)^{r+1} = (A(n)P(n) + R(n))^{r+1} = \sum_{i=0}^{r+1} \binom{r+1}{i} A(n)^i P(n)^i R(n)^{r+1-i}.$$

For $i = r+1$ and r it is immediate that $P(n)^r \mid \binom{r+1}{i} A(n)^i P(n)^i R(n)^{r+1-i}$.

And for each i with $1 \leq i \leq r-1$, since if n works for $r-1$ then it also works for everything less than $r-1$, the inductive assumption assures as that $P(n)^{r-i} \mid R(n)^{r+1-i}$, and so again $P(n)^r \mid \binom{r+1}{i} A(n)^i P(n)^i R(n)^{r+1-i}$. It follows that $P(n)^r \mid R(n)^{r+1}$, as desired.

Now suppose that R is not the zero polynomial. As $n \to \infty$ we have $P(x) \sim cx^t$ and $R(x) \sim dx^u$ where $t = \deg(P)$ and $u = \deg(R) \leq t-1$ and $c, d \neq 0$. It follows that

$$tr \leq u(r+1) \leq (t-1)(r+1) = tr - r + t - 1 \quad \Rightarrow \quad r \leq t-1.$$

However, if r were chosen so that $r \geq t$ at the outset, this would contradict the last inequality. Hence R is the zero polynomial and $P(x) \mid Q(x)$, as desired.

5. Determine all pairs (a, b) of real numbers that simultaneously satisfy the equations

$$a^{20} + b^{20} = 1 \qquad \text{and} \qquad a^{21} + b^{21} = 1.$$

Solution (Alice Devillers)

Since $a^{20} \geq 0$ and $b^{20} \geq 0$, we have $a^{20} = 1 - b^{20} \leq 1$ and similarly $b^{20} \leq 1$. It follows that $a \leq 1$ and $b \leq 1$.

Hence we have $a^{20}(a - 1) \leq 0$ and $b^{20}(b - 1) \leq 0$, which are equivalent to $a^{21} \leq a^{20}$ and $b^{21} \leq b^{20}$. Therefore, to have $a^{20} + b^{20} = a^{21} + b^{21} = 1$, we must have $a^{20} = a^{21}$ and $b^{20} = b^{21}$.

The equation $a^{20} = a^{21}$ is equivalent to $a^{20}(a - 1) = 0$, so it implies that $a = 0$ or $a = 1$. Similarly, we have $b = 0$ or $b = 1$.

This leaves four cases to check and the only solutions among these are $(a, b) = (1, 0)$ and $(a, b) = (0, 1)$.

6. A school has 60 students in year 2 who will be divided into three classes of 20 students. Each student writes a list of three other students that they hope to have in their class.

   Can the school always arrange for each student to be in the same class as at least one of the three students on their list?

Solution 1 (Angelo Di Pasquale)

Answer: No

Consider the situation where universally popular students A, B and C are on everyone's list, except that A is not on A's list, B is not on B's list and C is not on C's list. Also suppose that a student D is on A's, B's and C's list. Thus each student now has three students on their lists. We will show that the school cannot fulfil its claim.

Suppose that one of the classes contains none of A, B, C. Then no one in that class gets any of their preferences.

Suppose that each of the three classes contains one of A, B or C. Without loss of generality we may suppose that D is in the same class as A. Then student B does not get any of their preferences.

Either way the school is unable to fulfil its claim.

Solution 2 (Michelle Chen)

Here's another situation where the school can't fulfil its claim. Split the students into two groups, $A = \{A_1, A_2, A_3, A_4\}$ and $B = \{B_1, B_2, \ldots, B_{56}\}$. Suppose each A student puts down the other three A students on their list, while each B student B_i puts down $B_{i+1}, B_{i+2}, B_{i+3}$ (with indices wrapping around mod 56) on their list.

Consider the class containing A_1. To make A_1 happy, one of A_2, A_3, A_4 must also be in this class, thus the number of B students in this class is at most $20 - 2 = 18$. Suppose the B students in this class are $B_{j_1}, B_{j_2}, \ldots, B_{j_k}$ with $j_1 < j_2 < \cdots < j_k$ and $1 \leq k \leq 18$. To ensure each of these B students are happy, we must have $j_2 - j_1 \leq 3$, $j_3 - j_2 \leq 3$, $\ldots$, $j_k - j_{k-1} \leq 3$ and $j_1 + 56 - j_k \leq 3$. However, adding these equations together gives $56 \leq 3k$, which contradicts $k \leq 18$.

Solution 3 (Andrew Elvey Price)

Consider the situation where there are 50 special students named $s_1, s_2, \ldots, s_{50}$. For each j, the student s_j chooses s_{j+1}, s_{j+2} and s_{j+3}, where $s_{j+50} = s_j$. The other 10 students can choose anyone. Since 20 students must be in each class, there must be at least some special student s_i in each class. We know that if s_j is in a certain class then one of s_{j+1}, s_{j+2} and s_{j+3} is in the same class, so the gap $(k - j)$ from any student s_j to the next student s_k in the same class is at most 3. Since the sum of all such gaps for a given class is 50, there are at least $50/3$ special students in each class, that is, at least 17. But this is impossible as there are only 50 special students in total, less than $17 \times 3 = 51$.

7. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 1$ and for $n = 1, 2, 3, \ldots$

$$a_{n+1} = a_n^2 + 1.$$

Prove that there exists a positive integer n such that a_n has a prime factor with more than 2021 digits.

Solution 1 (Angelo Di Pasquale)

Call a prime p *good* if there exists a positive integer n such that a_n is divisible by p. It suffices to show that there are infinitely many good primes.

For a good prime p, let d be the smallest positive integer such that $a_d \equiv 0 \pmod p$. Define $a_0 = 0$. An easy induction yields $a_i \equiv a_{i+d} \pmod p$ for all integers $i \geq 0$. So $a_n \equiv 0 \pmod p$ whenever $d \mid n$.

Suppose, for the sake of contradiction, that there are only finitely many good primes $p_1, p_2, \ldots, p_k$ and let d_i denote the smallest positive integer such that $a_{d_i} \equiv 0 \pmod{p_i}$. From the preceding paragraph we know that $a_n \equiv 0 \pmod{p_i}$ whenever $d_i \mid n$. Choose $n = d_1 d_2 \cdots d_k$. Hence a_n is divisible by $p_1 p_2 \cdots p_k$. Let p be a prime factor of a_{n+1}. Hence p is good and so is in the list $p_1, p_2, \ldots, p_k$. But $a_{n+1} = a_n^2 + 1$ and $p \mid a_{n+1}$ and $p \mid a_n$. Thus $p \mid 1$, which is a contradiction.

Solution 2 (Daniel Mathews)

We claim that for each prime p, there is a positive integer d such that no a_n is divisible by p^d. In other words, the number of times p divides any a_n is bounded above by d.

Let a_m be the first term in the sequence divisible by p. Suppose p divides into a_m precisely k times; that is, $p^k \mid a_m$ but p^{k+1} does not divide a_m. We then consider the sequence a_n modulo p^k and p^{k+1}.

Modulo p^k, we have $a_1 = 1$, then all a_n are nonzero for $2 < n < m - 1$, then $a_m = 0$. After this, $a_{m+1} = 1$, and the sequence must repeat with period m.

Modulo p^{k+1}, we also have $a_1 = 1$. For $2 < n < k - 1$ all a_n are again nonzero. By definition of k we have a_m nonzero. Now since $p^k \mid a_m$ we have $p^{k+1} \mid a_m^2$ (in fact $p^{2k} \mid a_m^2$), so $a_m^2 = 0 \bmod p^{k+1}$, hence $a_{m+1} = a_m^2 + 1 = 1 \bmod p^{k+1}$. Thus $a_{m+1} = a_1 \bmod p^{k+1}$ and the sequence is again periodic with period m. As all terms from a_1 to a_m are nonzero mod p^{k+1}, we see that for any positive integer n, a_n is not divisible by p^{k+1}. This proves the claim (by taking $d = k + 1$).

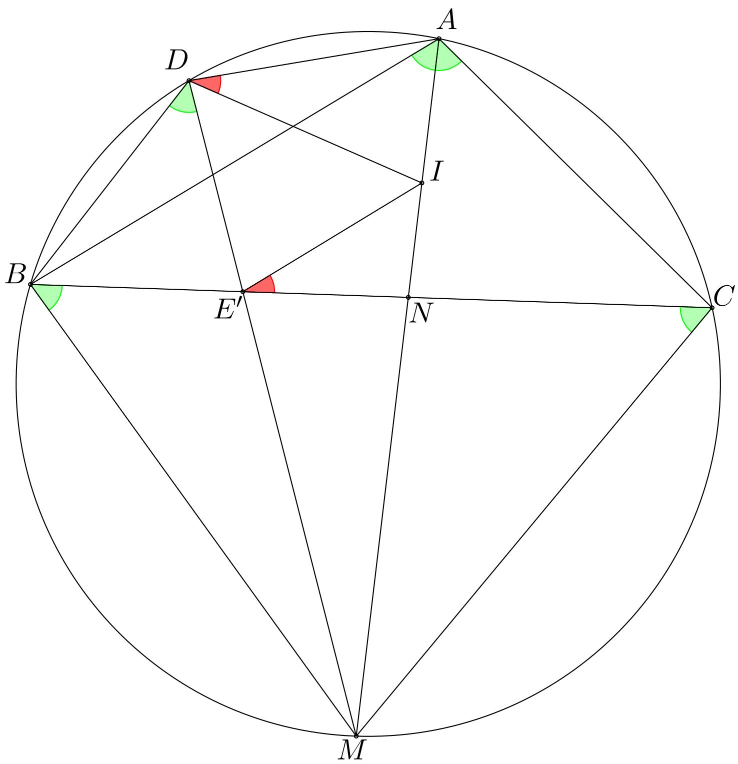
Now suppose there are only finitely many primes which divide terms of the sequence; call them $p_1, \ldots, p_N$. By the claim, there are integers $d_1, \ldots, d_N$ such that the number of times p_i divides any a_n is at most d_i. Hence every term in the sequence is of the form $p_1^{c_1} \cdots p_N^{c_N}$ where each c_i satisfies $0 \leq c_i \leq d_i$. Hence only finitely many distinct integers appear in the sequence. But the sequence a_n is clearly increasing and hence contains infinitely many distinct integers. This contradiction gives the desired result.

8. Let ABC be a triangle with incentre I. Suppose that D is a variable point on the circumcircle of ABC, on the arc AB that does not contain C. Let E be a point on the line segment BC such that $\angle ADI = \angle IEC$.

Prove that, as D varies, the line DE passes through a fixed point.

Solution (Sampson Wong)

Let AI meet the circumcircle of ABC again at M. We claim M is the fixed point. Let E' be the intersection of DM and BC. We will show that $\angle ADI = \angle IE'C$, which implies $E = E'$ since $\angle IEC$ varies monotonically as E varies along BC.



First, note that $MB = MC = MI$. Certainly $MB = MC$ holds since arcs MB and MC subtend equal angles at the circumference. The equality $MB = MI$ follows from

$$\angle MBI = \angle MBC + \angle CBI = \angle MAC + \angle IBA = \angle MAB + \angle IBA = \angle MIB.$$

Next, let AM intersect BC at N. Since $\angle CBM = \angle MAB = \angle MDB$, we have the similarities $\triangle MBE' \sim \triangle MDB$ and $\triangle MBN \sim \triangle MAB$. These imply the following length conditions:

$$MI^2 = MB^2 = MD \times ME' = MA \times MN.$$

The condition $MI^2 = MD \times ME'$ implies that $\triangle MIE' \sim \triangle MDI$, while the condition $MD \times ME' = MA \times MN$ implies that $\triangle MNE' \sim \triangle MDA$. Finally, the proof can be completed by noting

$$\angle ADI = \angle ADM - \angle IDM = \angle MNE' - \angle MIE' = \angle IE'C.$$

Remark The similar triangles which allowed the conversions between angle and length conditions can be replaced by power of point arguments, or an inversion with centre M and radius $MB = MC = MI$.

Score Distribution/Problem

Mark/problem	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8
0	2	15	90	112	6	61	81	125
1	1	26	3	9	17	5	2	8
2	12	9	6	6	5	1	12	3
3	13	4	37	7	2	0	8	0
4	10	5	6	2	1	0	0	0
5	13	4	3	7	9	2	3	0
6	11	3	3	2	27	9	1	1
7	98	94	12	15	93	82	53	23
Average	**5.8**	**4.8**	**1.7**	**1.3**	**5.6**	**4.0**	**2.8**	**1.1**