

DAY 1

Wednesday, 2 February 2022

Time allowed: 4 hours

No calculators are to be used.

Each question is worth seven points.

1. Prove that a convex pentagon with integer side lengths and an odd perimeter can have two right angles, but cannot have more than two right angles.
(A polygon is *convex* if all of its interior angles are less than 180° .)
2. Suppose that P is a point inside a convex hexagon $ABCDEF$ such that $PABC$, $PCDE$ and $PEFA$ are parallelograms of equal area.
Prove that $PBCD$, $PDEF$ and $PFAB$ are also parallelograms of equal area.
3. Determine all polynomials $p(x)$ with real coefficients such that:
 - $p(x) > 0$ for all positive real numbers x
 - $\frac{1}{p(x)} + \frac{1}{p(y)} + \frac{1}{p(z)} = 1$ for all positive real numbers x, y, z satisfying $xyz = 1$.
4. Let S be the set of points (i, j) in the plane with $i, j \in \{1, 2, \dots, 2022\}$ and $i < j$. We would like to colour each point in S either red or blue, so that whenever points (i, j) and (j, k) have the same colour, the point (i, k) also has that colour.
How many such colourings of S are there?

DAY 2

Thursday, 3 February 2022

Time allowed: 4 hours

No calculators are to be used.

Each question is worth seven points.

5. Let $x_1, x_2, \dots, x_{2022}$ be a sequence of real numbers such that $x_1 + x_2 + \dots + x_{2022} > 0$. Suppose that there exists a positive integer k less than 2022 such that:

- the first k terms of the sequence are less than or equal to zero
- the remaining terms of the sequence are greater than or equal to zero.

Prove that $x_1 + 2x_2 + 3x_3 + \dots + 2022x_{2022} > 0$.

6. In the country of Biplania there are two airline companies: Redwing and Blueways. There are 20 cities in Biplania, including Abadu and Ethora. Between any pair of cities, exactly one of the two airlines has direct flights and that airline flies in both directions between the two cities.

Arthur only flies on Redwing. He can travel from Abadu to Ethora in exactly four flights but not fewer.

Martha only flies on Blueways.

Show that Martha can travel between any two cities in Biplania in at most two flights.

7. For any positive integer n , let $f(n)$ denote the absolute value of the difference between n and its nearest power of 2. We define the *stopping time* of n to be the smallest positive integer k such that $f(f(\dots f(n)\dots)) = 0$, where f is applied k times. For example, $f(54) = 10$, $f(10) = 2$, and $f(2) = 0$. So $f(f(f(54))) = 0$ and the stopping time of 54 is 3.

- (a) What is the smallest positive integer whose stopping time is 2022?
- (b) What is the maximum number of consecutive positive integers that have the same stopping time?

8. Let ABC be an acute scalene triangle with orthocentre H . The altitudes from B and C intersect the angle bisector of $\angle BAC$ at D and E respectively. The tangents to the circumcircle of triangle DEH at D and E intersect at X .

Prove that $BX = CX$.