AMO. 2022 AUSTRALIAN MATHEMATICAL OLYMPIAD

DAY 1
Wednesday, 2 February 2022
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Prove that a convex pentagon with integer side lengths and an odd perimeter can have two right angles, but cannot have more than two right angles.

   (A polygon is *convex* if all of its interior angles are less than 180°.)

2. Suppose that P is a point inside a convex hexagon $ABCDEF$ such that $PABC$, $PCDE$ and $PEFA$ are parallelograms of equal area.

   Prove that $PBCD$, $PDEF$ and $PFAB$ are also parallelograms of equal area.

3. Determine all polynomials $p(x)$ with real coefficients such that:

   - $p(x) > 0$ for all positive real numbers x
   - $\dfrac{1}{p(x)} + \dfrac{1}{p(y)} + \dfrac{1}{p(z)} = 1$ for all positive real numbers x, y, z satisfying $xyz = 1$.

4. Let S be the set of points (i, j) in the plane with $i, j \in \{1, 2, \dots, 2022\}$ and $i < j$. We would like to colour each point in S either red or blue, so that whenever points (i, j) and (j, k) have the same colour, the point (i, k) also has that colour.

   How many such colourings of S are there?

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The Olympiad program is supported by the Australian Government Department of Industry, Science, Energy and Resources through the Science Competitions: Mathematics and Informatics Olympiads grant opportunity.

DAY 2
Thursday, 3 February 2022
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. Let $x_1, x_2, \ldots, x_{2022}$ be a sequence of real numbers such that $x_1 + x_2 + \cdots + x_{2022} > 0$. Suppose that there exists a positive integer k less than 2022 such that:

   - the first k terms of the sequence are less than or equal to zero
   - the remaining terms of the sequence are greater than or equal to zero.

   Prove that $x_1 + 2x_2 + 3x_3 + \cdots + 2022x_{2022} > 0$.

6. In the country of Biplania there are two airline companies: Redwing and Blueways. There are 20 cities in Biplania, including Abadu and Ethora. Between any pair of cities, exactly one of the two airlines has direct flights and that airline flies in both directions between the two cities.

   Arthur only flies on Redwing. He can travel from Abadu to Ethora in exactly four flights but not fewer.

   Martha only flies on Blueways.

   Show that Martha can travel between any two cities in Biplania in at most two flights.

7. For any positive integer n, let $f(n)$ denote the absolute value of the difference between n and its nearest power of 2. We define the *stopping time* of n to be the smallest positive integer k such that $f(f(\cdots f(n)\cdots)) = 0$, where f is applied k times. For example, $f(54) = 10$, $f(10) = 2$, and $f(2) = 0$. So $f(f(f(54))) = 0$ and the stopping time of 54 is 3.

   (a) What is the smallest positive integer whose stopping time is 2022?

   (b) What is the maximum number of consecutive positive integers that have the same stopping time?

8. Let ABC be an acute scalene triangle with orthocentre H. The altitudes from B and C intersect the angle bisector of $\angle BAC$ at D and E respectively. The tangents to the circumcircle of triangle DEH at D and E intersect at X.

   Prove that $BX = CX$.

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1. Prove that a convex pentagon with integer side lengths and an odd perimeter can have two right angles, but cannot have more than two right angles.

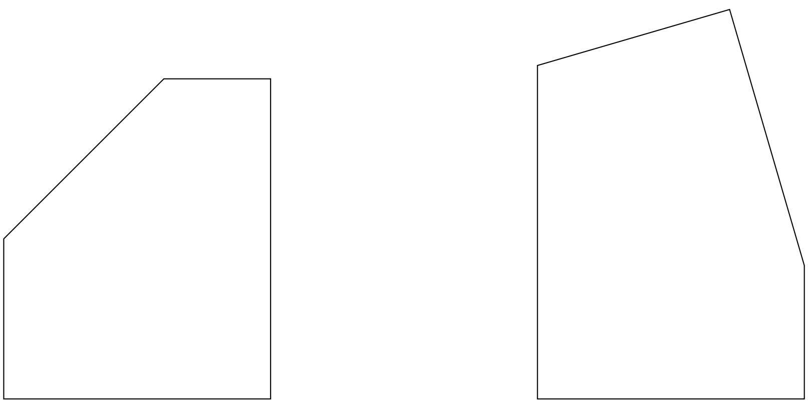
   (A polygon is *convex* if all of its interior angles are less than 180°.)

Solution (Mike Clapper)

Firstly we establish that such a pentagon with two right angles is possible. A unit equilateral triangle placed on top of a unit square forms a pentagon with perimeter 5.

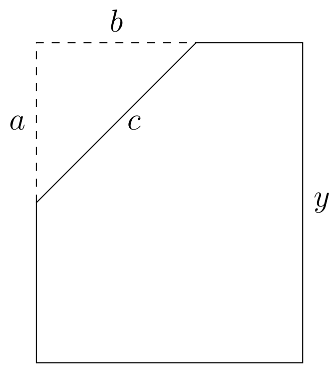
From the angle sum formula, we see that if a pentagon has four right angles, the fifth angle would be 180° which means that the pentagon actually degenerates into a rectangle (and would have an even perimeter anyway).

So it remains to show that a pentagon with integer side lengths and three right angles must have an even perimeter. If three out of five angles are right angles, then at least two of them must be adjacent. This gives two cases, all three right angles are adjacent or only two are adjacent, as shown below:



Case 1 Case 2

Firstly consider Case 1:

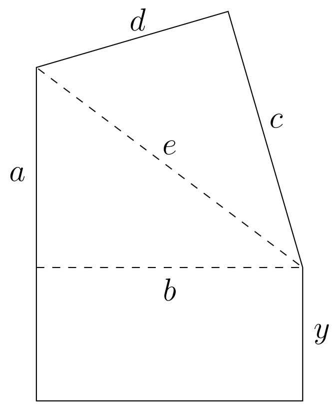


The bounding rectangle has integer sides and hence its perimeter has even parity. Furthermore, the dotted section forms a right-angled triangle with integer sides. The perimeter of the pentagon is found by replacing the two dotted sides with the hypotenuse. But since

$a^2 + b^2 = c^2$ implies that $a + b$ has the same parity as c, the perimeter of the pentagon is also even.

Algebraically, using the labelling in the diagram, the perimeter of the bounding rectangle is $2(x + y)$ and the perimeter of the pentagon is $2(x + y) + (c - a - b)$.

Now consider Case 2:



Draw a line parallel to the base as in the diagram to form two right-angled triangles, both with hypotenuse e. We see that $a^2 + b^2 = c^2 + d^2$. This means that $a + b$ has the same parity as $c+d$ and hence $a+b+c+d$ is even. As the perimeter is given by $a+b+c+d+2y$ it follows that the pentagon has even perimeter.

Hence a pentagon with integer sides and an odd perimeter can have at most two right angles.

2. Suppose that P is a point inside a convex hexagon $ABCDEF$ such that $PABC$, $PCDE$ and $PEFA$ are parallelograms of equal area.

   Prove that $PBCD$, $PDEF$ and $PFAB$ are also parallelograms of equal area.

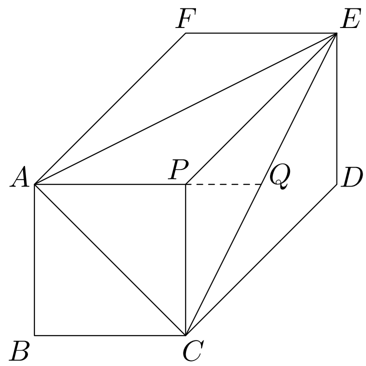
Solution 1 (Angelo Di Pasquale)

A diagonal of a parallelogram divides it into two triangles of equal area. Hence the lines connecting P to the vertices of the hexagon partition into six triangles of equal area. From this it follows that $PBCD$, $PDEF$ and $PFAB$ are quadrilaterals of equal area.

We also have

$$|APC| = \frac{1}{2}|PABC| = \frac{1}{2}|PEFA| = |APE|.$$

Let line Q be the intersection of lines AP and CE.



We have

$$\frac{|CPQ|}{|EPQ|} = \frac{CQ}{QE} = \frac{|CAQ|}{|EAQ|}.$$

Using addendo this implies

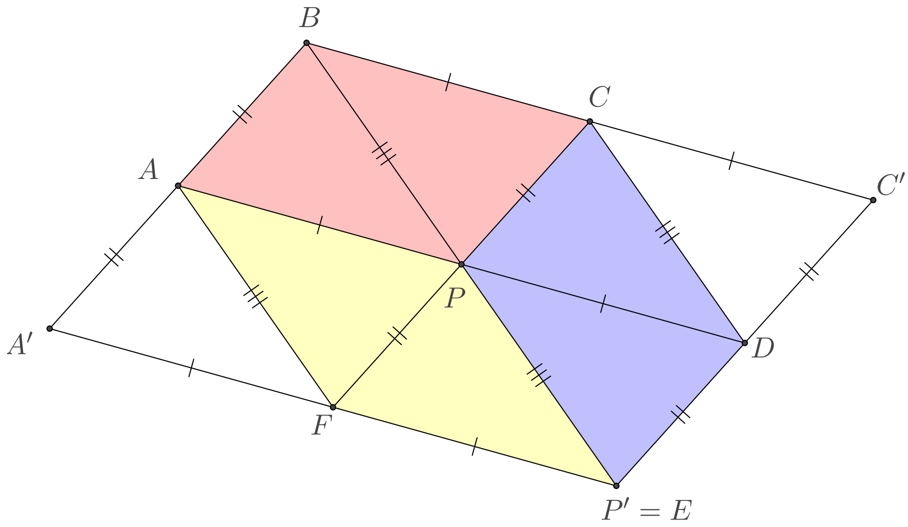
$$\frac{CQ}{QE} = \frac{|CAQ| - |CPQ|}{|EAQ| - |EPQ|} = \frac{|APC|}{|APE|} = 1.$$

Thus Q is the midpoint of CE.

Since $PCDE$ is a parallelogram, its diagonals bisect each other. In particular PQD is a straight line. Hence points A, P, Q, and D are collinear. Thus $PD \parallel FE$. A similar argument shows that $PF \parallel DE$. Hence $PDEF$ is a parallelogram. Similarly $PFAB$ and $PBCD$ are also parallelograms, as required.

Solution 2 (Thanom Shaw)

Consider the dilation by a factor of 2 with centre B and let $BA'P'C'$ be the image of parallelogram $BAPC$.



Since $PABC$ and $PEFA$ are parallelograms of equal area, side EF lies on the line $P'A'$. Since $PABC$ and $PCDE$ are parallelograms of equal area, side ED lies on the line $P'C'$. Since E lies on both $P'A'$ and $P'C'$, E is the point of intersection of these lines. That is, $E = P'$ and P is the midpoint of BE. Similarly, P is also the midpoint of AD and CF.

Now PB, PD, and PF divide parallelograms $PABC$, $PCDE$, and $PEFA$ into congruent triangles PAB, BCP, DPC, PDE, EFP, APF, of equal area. Note that corresponding sides of these triangles are equal given opposite sides of parallelograms are equal and P is the midpoint of BE, AD and CF.

Finally, we have that each of $PBCD$, $PDEF$ and $PFAB$ are parallelograms as their opposite sides are equal. Furthermore, each parallelogram consists of the pairs of congruent triangles of equal area, BCP & DPC, PDE & EFP, APF & PAB. Hence $PBCD$, $PDEF$ and $PFAB$ are parallelograms of equal area.

Solution 3 (Angelo Di Pasquale)

The problem is invariant under an affine transformation. Hence we may assume without loss of generality that $PABC$ is a unit square, with $B = (0,0)$, $C = (1,0)$, $P = (1,1)$, and $A = (0,1)$. From $|PABC| = |PEFA|$, point E lies on the line $y = 2$. From $|PABC| = |PCDE|$, point E lies on the line $x = 2$. Hence $E = (2,2)$. Since $\overrightarrow{AP} = \overrightarrow{FE}$, we have $F = (1,2)$, and similarly $D = (2,1)$. From these, the conclusion easily follows.

Solution 4 (Chaitanya Rao)

Establish that the quadrilaterals have equal area as in Solution 1. To show they are parallelograms, we start by writing the given equal area condition as equal vector cross products: $\overrightarrow{PA} \times \overrightarrow{PC} = \overrightarrow{PC} \times \overrightarrow{PE} = \overrightarrow{PE} \times \overrightarrow{PA}$. From the first equality,

$$\overrightarrow{PA} \times \overrightarrow{PC} = -\overrightarrow{PE} \times \overrightarrow{PC} \quad \Longrightarrow \quad (\overrightarrow{PA} + \overrightarrow{PE}) \times \overrightarrow{PC} = \overrightarrow{0}.$$

Since $PEFA$ is a parallelogram, $\overrightarrow{PA} + \overrightarrow{PE} = \overrightarrow{PA} + \overrightarrow{AF} = \overrightarrow{PF}$ so $\overrightarrow{PF} \times \overrightarrow{PC} = \overrightarrow{0}$. As the two vectors are each non-zero, we conclude that $FP \parallel PC \parallel AB$. By a similar argument we can show that $BP \parallel PE \parallel AF$. Hence $PFAB$ is a parallelogram. Similarly $PBCD$ and $PDEF$ are parallelograms.

3. Determine all polynomials $p(x)$ with real coefficients such that:

- $p(x) > 0$ for all positive real numbers x
- $\dfrac{1}{p(x)} + \dfrac{1}{p(y)} + \dfrac{1}{p(z)} = 1$ for all positive real numbers x, y, z satisfying $xyz = 1$.

Solution 1 (Norman Do)

Substituting $x = y = z = 1$ yields $P(1) = 3$. Now using the given relation, we obtain

$$\frac{1}{P(x)} + \frac{1}{P(x)} + \frac{1}{P(x^{-2})} = \frac{1}{P(x^2)} + \frac{1}{P(1)} + \frac{1}{P(x^{-2})}.$$

This implies that

$$\frac{2}{P(x)} = \frac{1}{P(x^2)} + \frac{1}{3}.$$

Now multiply out to obtain the equation

$$6P(x^2) = 3P(x) + P(x^2)P(x).$$

This must be true for all $x > 0$, which then implies that the equation is true at the level of polynomials. If the degree of $P(x)$ is $d > 0$, then the left side of this equation has degree $2d$, while the right side has degree $3d$. This yields a contradiction, so it must be the case that $P(x)$ is a constant polynomial. Since $P(1) = 3$, the only possible solution is $P(x) = 3$, which does indeed satisfy the given conditions.

Solution 2 (William Steinberg)

We first show that if P is a polynomial satisfying similar conditions in two variables for $xy = 1$ then $P(x) = x^n + 1$ for some n. This means that for $xy = 1$ we have

$$\frac{1}{P(x)} + \frac{1}{P(y)} = 1$$

which we can rewrite as $(P(x) - 1)(P(x^{-1}) - 1) = 1$. Define $Q(x) = P(x) - 1$ and $R(x) = x^d Q(x^{-1})$ where d is the degree of Q. In other words, R is the same polynomial as Q but with coefficients reversed. Thus for each positive real x we have that $Q(x)R(x) = x^d$, so both Q and R must be monomials and hence $P(x) = Q(x) + 1 = cx^n + 1$ for some c and n. Substituting this back into the original condition and expanding we find that this works when $c = \pm 1$, but $c = -1$ violates the condition that $P(x) > 0$ for $x > 0$.

Now we go back to the original problem. Substitute $x = y = z = 1$ to obtain $P(1) = 3$. Substituting $z = 1$ gives us, for $xy = 1$,

$$\frac{1}{P(x)} + \frac{1}{P(y)} = 1 - \frac{1}{P(1)} = \frac{2}{3}.$$

Observe that $P'(x) = \frac{2}{3}P(x)$ satisfies the same conditions as in the two-variable version described so $P'(x) = x^n + 1$ for some n and hence $P(x) = \frac{3}{2}x^n + \frac{3}{2}$. If $n > 0$ then taking $x = y = 3^{-\frac{1}{n}}$ and $z = 9^{\frac{1}{n}}$ yields a contradiction. Thus $P(x) = 3$ and this clearly works.

Solution 3 (Ian Wanless)

Suppose first that $P(x)$ is not constant. Rewriting the equation we have

$$1-\frac{1}{P(x)}=\frac{1}{P(y)}+\frac{1}{P(1/(xy))}$$

whenever $x, y > 0$. Fixing x but taking the limit as $y \to \infty$ we see the right-hand side approaches $1/P(0)$. But this is independent of x, so the left-hand side must be $1/P(0)$ too. Since $P(x)$ is constant on $x > 0$ it must be constant. It's clear that if $P(x)$ is constant then it must be that $P(x) = 3$, and that this is a solution.

4. Let S be the set of points (i, j) in the plane with $i, j \in \{1, 2, \ldots, 2022\}$ and $i < j$. We would like to colour each point in S either red or blue, so that whenever points (i, j) and (j, k) have the same colour, the point (i, k) also has that colour.

   How many such colourings of S are there?

Solution 1 (Ivan Guo)

The answer is 2022!. We will work with the general case of $S = \{(i, j) : 1 \leq i < j \leq N\}$ and present a bijection between valid colourings and permutations on $1, \ldots, N$.

First, construct a map from permutations to colourings as follows. Given a permutation $(a_1, a_2, \ldots, a_N)$, colour (i, j) red if $a_i < a_j$ and blue otherwise. The colouring satisfies the required conditions since:

- If (i, j) and (j, k) are both red, then $a_i < a_j < a_k$. Hence (i, k) is also red.
- If (i, j) and (j, k) are both blue, then $a_i > a_j > a_k$. Hence (i, k) is also blue.

Next, we show that this map is injective. If two permutations $(a_i)_{1 \leq i \leq N}$, $(b_i)_{1 \leq i \leq N}$ map to the same colouring, then for every $i < j$, we have $a_i < a_j$ if and only if $b_i < b_j$. Hence, for every i, the sets $\{j : a_j > a_i\}$ and $\{j : b_j > b_i\}$ are identical. Thus the two permutations must be the same.

It remains to check that the map is surjective. We will prove that every colouring is obtainable from this mapping via an induction on N. The idea is to identifying a number p such that (k, p) is red for every $k < p$, and (p, k) is blue for every $k > p$. If this can be done then a_p must be N in the permutation, while the other elements can be reduced to the $N - 1$ case. The base case of $N = 1$ is trivially true.

Define the sequence $c_1, c_2, \ldots, c_m$ as follows. Let $c_1 = 1$. For each $i > 1$, let c_i be the smallest integer such that $c_i > c_{i-1}$ and (c_{i-1}, c_i) is red. The construction continues for as long as possible, and terminates at $c_m = p$. Consequently, (p, k) must be blue for all $k > p$.

We claim that for all $k < p$, (k, p) is red. Since the colouring is valid, (c_i, c_j) must be red for all $i < j$. Hence (c_i, p) is red for all $c_i < p$. Suppose there is some q such that $c_i < q < c_{i+1}$ and (q, p) is blue. By the minimality of c_{i+1}, (c_i, q) must be blue. This implies that (c_i, p) is blue, which is a contradiction. So the required property on p is proven.

To complete the induction, we remove all points in S which have coordinates involving p, and consider the colouring of the remaining points $\{(i, j) : i \neq p, j \neq p, 1 \leq i < j \leq N\}$. This is equivalent to a colouring of the $N - 1$ case. By the inductive hypothesis, there is permutation $(b_1, \ldots, b_{N-1})$ of $1, \ldots, N - 1$ which maps to this colouring. It is then straightforward to check that the permutation $(b_1, \ldots, b_{p-1}, N, b_p, \ldots, b_{N-1})$ maps to our original colouring. This establishes the bijection and completes the proof.

Solution 2 (Angelo Di Pasquale)

Let G_n be the directed graph on n vertices $V_1, \ldots V_n$ such that $V_i \to V_j$ if (i, j) is red and $V_j \to V_i$ if (i, j) is blue. Note:

(i) For each $P, Q \in G_n$ with $P \neq Q$, we have exactly one of $P \to Q$ or $Q \to P$.

(ii) If $A \to B$ and $B \to C$, then $A \to C$.

A complete directed graph satisfying (i) and (ii) shall be called *nice*. There's clearly a bijection between valid colourings and nice complete directed graphs with labelled vertices.

We wish to count the number of nice graphs on n labelled vertices. We prove by induction that the answer is $n!$ for each $n \geq 2$.

The base case is immediate.

Assume the answer is $n!$ for all nice graphs on n labelled vertices. Let G_{n+1} be a nice graph on $n+1$ labelled vertices.

Pick any point P in G_{n+1}, and starting from this point walk around following the arrows. If we get back to P, then let Q be the last point on the walk just before P so that $Q \to P$. But $P \to Q$ due to multiple applications of (ii). This contradicts (i). Hence we never get back to P. Since the graph is finite, the walk must terminate at some vertex, Z say. Note that Z has only in-edges. This implies that no other vertex Z' has only in-edges because otherwise both $Z \to Z'$ and $Z' \to Z$ would both occur, which contradicts (i).

Hence we have shown that there is exactly one vertex, Z say, in G_{n+1} having only in-edges.

If we remove Z from G_{n+1}, we get a nice graph on n labelled vertices. Thus every nice graph on $n+1$ labelled vertices is obtainable from a nice graph on n vertices by adding a single vertex Z which has only in-edges. Conversely, every such extension of a nice graph on n vertices results in a nice graph on $n+1$ vertices.

Since there are $n+1$ ways to choose Z in G_{n+1} and $n!$ ways of completing a nice graph on the remaining n labelled vertices, it follows that there are $(n+1)n! = (n+1)!$ nice graphs on $n+1$ labelled vertices.

Solution 3 (Alan Offer)

For a colouring of S, define the relation '$\prec$' on the set $N = \{1, 2, \ldots, 2022\}$ by:

- $i \prec i$ (so $\prec$ is reflexive)
- for $i < j$, $i \prec j$ if (i, j) is blue, while $j \prec i$ if (i, j) is red (so $\prec$ is antisymmetric and, further, every pair of points is comparable).

We will show that the given condition on the colouring of S is then equivalent to the condition that $\prec$ is transitive. Suppose that $i \prec j$ and $j \prec k$. There are six cases:

(a) If $i < j < k$, (i, j) and (j, k) are both blue. Then (i, k) is coloured validly (blue) if and only if $i \prec k$.

(b) If $i < k < j$, (i, j) is blue and (j, k) is red. Then (i, k) is coloured validly (blue) if and only if $i \prec k$.

(c) If $j < i < k$, (i, j) is red and (j, k) is blue. Then (i, k) is coloured validly (blue) if and only if $i \prec k$.

(d) If $j < k < i$, (i, j) is red and (j, k) is blue. Then (i, k) is coloured validly (red) if and only if $i \prec k$.

(e) If $k < i < j$, (i, j) is blue and (j, k) is red. Then (i, k) is coloured validly (red) if and only if $i \prec k$.

(f) If $k < j < i$, (i, j) and (j, k) are both red. Then (i, k) is coloured validly (red) if and only if $i \prec k$.

Therefore $\prec$ is transitive if and only if all such triple of points have valid colourings. Thus $\prec$ is a total order.

Similarly, notice that a total order $\prec$ in turn determines an allowable colouring of S.

Thus the allowable colourings are in one-to-one correspondence with the total orders of N, which in turn correspond to the ways of writing the 2022 elements of N in a row, or in other words, the permutations of N. Thus the number of colourings is $2022!$.

5. Let $x_1, x_2, \ldots, x_{2022}$ be a sequence of real numbers such that $x_1 + x_2 + \cdots + x_{2022} > 0$. Suppose that there exists a positive integer k less than 2022 such that:

   - the first k terms of the sequence are less than or equal to zero
   - the remaining terms of the sequence are greater than or equal to zero.

   Prove that $x_1 + 2x_2 + 3x_3 + \cdots + 2022x_{2022} > 0$.

Solution 1 (Kevin McAvaney)

We prove the result for n terms. If all terms are non-negative then the result clearly holds. If there exists at least one negative term, then we may choose $1 \leq k < n$. Hence

$$\begin{aligned}
&x_1 + \cdots + (k-1)x_{k-1} + kx_k + (k+1)x_{k+1} + \cdots + nx_n \\
&> x_1 + \cdots + (k-1)x_{k-1} - k(x_1 + \cdots + x_{k-1} + x_{k+1} + \cdots + x_n) + (k+1)x_{k+1} + \cdots + nx_n \\
&= (1-k)x_1 + \cdots + (-1)x_{k-1} + x_{k+1} + \cdots + (n-k)x_n \\
&> 0.
\end{aligned}$$

Solution 2 (Ivan Guo)

For each $i = 1, \ldots, n$, define the partial sum $S_i = x_i + \cdots + x_n$. Note that the sequence $S_1, \ldots, S_n$ is non-decreasing up to S_{k+1} and then non-increasing. Since S_1 is positive and $S_n = x_n$ is non-negative, S_i must be non-negative for all i. Thus

$$x_1 + 2x_2 + \cdots + nx_n = S_1 + \cdots + S_n > 0.$$

Solution 3 (Angelo Di Pasquale)

Let $S = x_1 + 2x_2 + \cdots + 2022x_{2022}$.

Suppose that $x_i < 0 < x_j$. Then $i \leq k \leq j$. Replace x_i and x_j by $x_i + \varepsilon$ and $x_j - \varepsilon$ where $\varepsilon = \min\{|x_i|, |x_j|\}$ in S. The change in S (new minus old) is given by

$$-ix_i + i(x_i + \varepsilon) - jx_j + j(x_j - \varepsilon) = -(j-i)\varepsilon < 0.$$

So the new value of S is less than the old value.

By using a sequence of such replacements we can make 2021 of $x_1, \ldots, x_n$ equal to 0. At this point, the final value of S is positive. And since it is less than the initial value of S, we conclude that the initial value of S was also positive.

6. In the country of Biplania there are two airline companies: Redwing and Blueways. There are 20 cities in Biplania, including Abadu and Ethora. Between any pair of cities, exactly one of the two airlines has direct flights and that airline flies in both directions between the two cities.

   Arthur only flies on Redwing. He can travel from Abadu to Ethora in exactly four flights but not fewer.

   Martha only flies on Blueways.

   Show that Martha can travel between any two cities in Biplania in at most two flights.

Solution 1 (Ian Wanless)

We are dealing with a complete graph on 20 vertices (cities) in which every edge is coloured either red or blue, indicating which airline flies that route. We are told that the shortest red path from Abadu to Ethora has four edges. Let one such path involve cities A (Abadu), B, C, D and E (Ethora) in that order. The fact that this is a shortest path indicates that the edges AC, AD, AE, BD, BE and CE are all blue.

First notice that Martha can fly between any two cities in $\{A, B, C, D, E\}$ in at most two steps. If there is not a direct flight on Blueways then she must be trying to fly between consecutive cities on the path $ABCDE$. She can do this by flying via the end of the path furthest from the edge between her two cities.

So we can without loss of generality assume that Martha is trying to fly from city X which is not one of $\{A, B, C, D, E\}$. In order to not provide a shorter path for Arthur, we know that XA and XE cannot both be red. For the remainder of the proof, suppose without loss of generality that XA is blue. Hence Martha can reach C, D or E in two flights by flying via A. Suppose Martha can't reach B from X in at most two flights. Then XB must be red and XE must be red (otherwise XE followed by EB works). But now Arthur can fly AB, BX, XE which is a contradiction. It follows that Martha can reach any city in $\{A, B, C, D, E\}$ from X in at most 2 flights.

We may thus assume that Martha's destination is Y, where Y is not one of $\{A, B, C, D, E\}$ and XY is red. If AY is blue then Martha can fly via A, so we may assume that AY is red. To prevent Arthur from flying AY, YE we must have that YE is blue. But now if XE is blue then Martha can fly via E, so the only problem is if XE is red. But if XE is red then Arthur can fly AY, YX, XE. This contradiction finishes the proof.

Solution 2 (Michelle Chen)

Suppose Martha is trying to fly from city X to city Y. First, notice that AE must be blue since Arthur cannot fly directly from A to E. Now, there are three cases.

If $\{X, Y\} = \{A, E\}$, then Martha can fly directly from X to Y since AE is blue.

Suppose one of X or Y is in $\{A, E\}$. Without loss of generality, assume $X = A$ and $Y \neq E$. Since Arthur cannot fly AY, YE, at least one of AY and YE is blue. If AY is blue, then Martha can fly directly from X to Y. Otherwise, YE is blue and Martha can fly XE, EY.

Finally, we have the case where neither X nor Y is in $\{A, E\}$. If XY is blue, then Martha can fly directly from X to Y. Otherwise, suppose XY is red. Since Arthur cannot fly AX, XE, at least one of AX and XE is blue. Without loss of generality, say AX is blue. Now, if AY is blue, then Martha can fly XA, AY. Otherwise, if AY is red, then YE must be blue since Arthur cannot fly AY, YE. Also, XE must be blue since Arthur cannot fly AY, YX, XE. Hence, Martha can fly XE, EY.

7. For any positive integer n, let $f(n)$ denote the absolute value of the difference between n and its nearest power of 2. We define the *stopping time* of n to be the smallest positive integer k such that $f(f(\cdots f(n)\cdots)) = 0$, where f is applied k times. For example, $f(54) = 10$, $f(10) = 2$, and $f(2) = 0$. So $f(f(f(54))) = 0$ and the stopping time of 54 is 3.

   (a) What is the smallest positive integer whose stopping time is 2022?

   (b) What is the maximum number of consecutive positive integers that have the same stopping time?

Solution 1 (Angelo Di Pasquale)

(a) Answer: $\frac{1}{3}(2^{4043}+1)$.

For each positive integer i, let a_i denote the smallest positive integer with stopping time i. Observe that for any integer n satisfying $2^{k-1} \le n \le 2^k$ we have $f(n) \le 2^{k-2}$. It follows that

$$1 \le n \le 2^k \implies f(n) \le 2^{k-2}. \tag{1}$$

We shall prove $a_i = \frac{1}{3}(2^{2i-1}+1)$ by induction. The base cases $i = 1, 2$ are easily verified. Suppose that $a_i = \frac{1}{3}(2^{2i-1}+1)$ for $i \ge 2$. It is straightforward to verify that

$$2^{2i-3} < a_i < 2^{2i-2}.$$

Let b be any positive integer satisfying $b < 2^{2i-1} + a_i$. If $b \le 2^{2i-1}$, then from (1) we have $f(b) \le 2^{2i-3} < a_i$. If $2^{2i-1} < b < 2^{2i-1} + a_i$, then again $f(b) < a_i$. Either way, $f(b)$ has stopping time strictly less than i. Thus b has stopping time at most i. It follows that

$$a_{i+1} \ge 2^{2i-1} + a_i = 2^{2i-1} + \frac{2^{2i-1}+1}{3} = \frac{2^{2i+1}+1}{3}.$$

However, since

$$2^{2i-1} < \frac{2^{2i+1}+1}{3} \le 2^{2i-1} + 2^{2i-2},$$

we have

$$f\left(\frac{2^{2i+1}+1}{3}\right) = \frac{2^{2i+1}+1}{3} - 2^{2i-1} = \frac{2^{2i-1}+1}{3} = a_i.$$

Hence $\frac{1}{3}(2^{2i+1}+1)$ has stopping time $i+1$. From this we conclude that $a_{i+1} = \frac{1}{3}(2^{2i+1}+1)$, which finishes the induction.

(b) Answer: 3

For each positive integer n, let $S(n)$ denote its stopping time. Note $S(5) = S(6) = S(7) = 2$. So it is possible that three consecutive positive integers have the same stopping time. It remains to show that four consecutive positive integers cannot have the same stopping time.

Suppose, for the sake of contradiction, that some four consecutive positive integers have the same stopping time. Let n be the smallest integer such that $S(n) = S(n+1) = S(n+2) = S(n+3)$. Observe that $S(n) = 1$ if and only if n is a power of 2. Since no four consecutive positive integers can be powers of 2, there exists an integer $k \ge 3$ such that

$$2^k < n, n+1, n+2, n+3 < 2^{k+1}.$$

Furthermore we calculate that $S(2^k + 2^{k-1}) = 2$ and $S(2^k + 2^{k-1} \pm 1) = 3$. Hence none of the four consecutive integers can be $2^k + 2^{k-1}$. Thus it leaves us with two cases.

Case 1. $2^k < n, n+1, n+2, n+3 < 2^k + 2^{k-1}$

Then $f(n) = n-2^k$, $f(n+1) = n+1-2^k$, $f(n+2) = n+2-2^k$, and $f(n+3) = n+3-2^k$ are four smaller consecutive positive integers with the same stopping time. This contradicts the minimality of n.

Case 2. $2^k + 2^{k-1} < n, n+1, n+2, n+3 < 2^{k+1}$

This implies $f(n+3) = 2^{k+1} - n - 3$, $f(n+2) = 2^{k+1} - n - 2$, $f(n+1) = 2^{k+1} - n - 1$, and $f(n) = 2^{k+1} - n$ are four smaller consecutive positive integers with the same stopping time. This contradicts the minimality of n.

Hence no four consecutive positive integers can have the same stopping time.

Solution 2 (Ian Wanless)

(a) Write all numbers in binary. If n starts with 10 then $f(n)$ is obtained by removing the leading 1 from n. Otherwise n starts with 11, and in that case we complement every bit from the first 1 up to but not including the last 1 (we leave the last 1 as is). It follows that $f(n)$ is at least two bits shorter than n unless n is precisely of the form 11 followed by an arbitrary number of zeroes. It also follows that trailing zeros do not affect the stopping time.

Let a_m be the least integer with stopping time equal to m. It is simple to check that $a_1 = 1$ and $a_2 = 11$. Let $b_m = 1010 \cdots 1011$ (where there are m ones) for all $m > 1$. Since b_m has stopping time m, the definition of a_m implies $a_m \leq b_m$. We will show by induction that $a_m = b_m$ for all m. The base cases of $m = 1, 2$ clearly hold.

Suppose not and consider the smallest $m > 2$ for which $a_m < b_m$. By definition $f(b_m) = b_{m-1} = a_{m-1} \leq f(a_m)$. We will show this is incompatible with the characterisation above. If a_m has fewer bits than b_m then $f(a_m)$ will have fewer bits than $f(b_m) = b_{m-1}$, contradicting $f(b_m) \leq f(a_m)$. So a_m and b_m must have the same number of bits. But then $a_m < b_m$ means that a_m must begin with 10 and so $f(a_m)$ would remove only the leading 1. Then $a_m - f(a_m) = b_m - f(b_m)$, which contradicts $a_m - b_m < 0 \leq f(a_m) - f(b_m)$.

It follows that $a_{2022} = b_{2022} = 1 + \sum_{i=1}^{2021} 2^{2i-1} = (2^{4043} + 1)/3$.

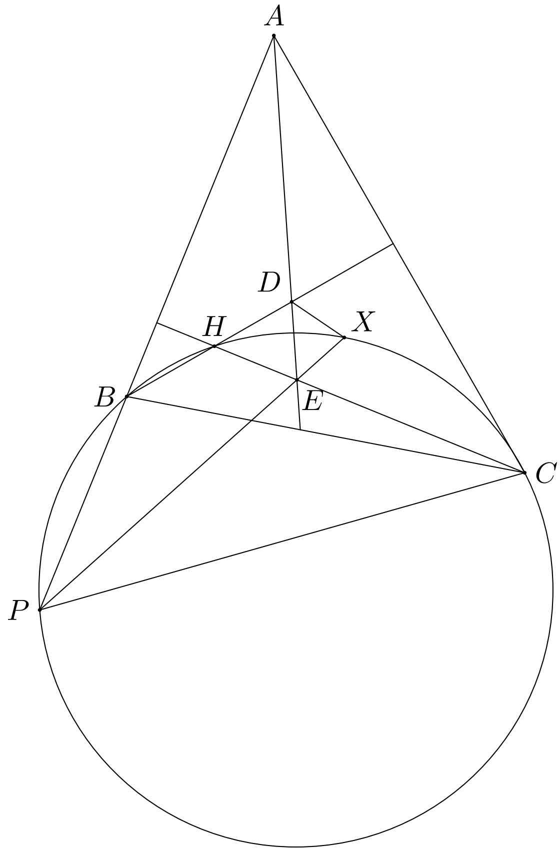
(b) Denote the stopping time of n by $S(n)$. Note that three consecutive numbers can have the same stopping time, $S(5) = S(6) = S(7) = 2$. To show that four consecutive numbers cannot have the same stopping time, it suffices to show that $S(4k) < S(4k \pm 1)$ for all $k \geq 1$.

We do this by induction. If k is a power of 2 then $S(4k) = 1 < 2 = S(4k \pm 1)$ as required. If $k = 3 \times 2^a$ for $a \geq 0$ then $S(4k) = 2 < 3 = S(4k \pm 1)$. For all other values of k, the numbers $f(4k-1), f(4k), f(4k+1)$ are three consecutive integers (in either increasing or decreasing order) and $f(4k)$ is divisible by 4 and less than $4k$. This reduces to an earlier case, so the claim follows by induction.

8. Let ABC be an acute scalene triangle with orthocentre H. The altitudes from B and C intersect the angle bisector of $\angle BAC$ at D and E respectively. The tangents to the circumcircle of triangle DEH at D and E intersect at X.

   Prove that $BX = CX$.

Solution 1 (Ivan Guo)



Let $\angle BAC = \alpha$. Then

$$\angle HED = \angle HCA + \angle CAE = 90^\circ - \alpha/2 = \angle HBA + \angle BAD = \angle HDE.$$

Hence $HE = HD$ and $HEXD$ is a kite with $\angle XDE = \angle XED = \angle DHE = \alpha$.

Next, reflect A about CH to obtain the point P. Then $\angle BPC = \alpha = 180^\circ - \angle BHC$, so $BHCP$ is cyclic. Since E lies on CH, the perpendicular bisector of AP, we have $\angle PEH = \angle AEH = \angle EDH$. So EP is tangent to the circumcircle of triangle DEH, and thus X, E and P are collinear.

Since $HEXD$ is a kite, we have

$$\angle HXP = 90^\circ - \angle DEX = 90^\circ - \alpha = \angle ACH = \angle PCH.$$

Hence X also lies on the circumcircle of $BHCP$. Finally, since X lies on the external angle bisector of $\angle BHC$, it is the midpoint of the arc BC that contains H. Therefore $XB = XC$.

Solution 2 (William Steinberg)

Let P be the second intersection of circle BHC with AB and M be the midpoint of arc BHC. Note that,

$$\angle BPC = \angle EHD = \angle BAC.$$

Since

$$\angle ACH = \angle HBA = \angle HCP,$$

CE bisects $\angle ACP$. Also AE bisects $\angle PAC$. So E is the incentre of triangle APC and PE bisects $\angle CPA$.

Hence P, M, E are collinear. Thus

$$\angle MED = \angle PAE + \angle EPA = \angle BAC = \angle EHD$$

and ME is tangent to circle HDE. Similarly MD is tangent to circle HDE. Therefore $X = M$ and hence $BX = BM = CM = CX$ as desired.

Score Distribution/Problem

Mark/Problem	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8
0	10	35	59	186	26	31	57	120
1	13	31	37	0	14	16	22	46
2	6	7	35	3	2	9	22	3
3	15	34	11	1	0	7	21	4
4	18	29	4	0	1	6	11	0
5	17	4	3	0	8	33	24	0
6	26	5	10	1	20	12	7	1
7	88	48	34	2	122	79	29	19
Average	**5.2**	**3.4**	**2.4**	**0.2**	**5.4**	**4.5**	**2.8**	**1.1**