AMO. 2023 AUSTRALIAN MATHEMATICAL OLYMPIAD

DAY 1
Tuesday, 7 February 2023
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Let n be a positive integer greater than 2. Mia is playing the following game. She writes the numbers $1, 2, 3, \ldots, n$ in some order on the sides of a regular n-sided polygon, one number per side. Then, on each vertex of the polygon, she writes the sum of the numbers on the two sides that meet at that vertex. Mia wins if the n numbers on the vertices can be written down in some order to form an arithmetic progression.

   For which n can Mia win this game?

   (In an *arithmetic progression*, there is a constant d such that each term is equal to the previous term plus d.)

2. A set of integers is *wanless* if the sum of its elements is 1 less than a multiple of 4.

   How many subsets of $\{1, 2, 3, \ldots, 2023\}$ are wanless?

3. Let $f(x)$ be a polynomial with real coefficients not all equal to zero.

   Prove that there exists another polynomial $g(x)$ with real coefficients such that the polynomial $f(x)g(x)$ has exactly 2023 more positive coefficients than negative coefficients.

4. Let ABC be an acute triangle. Points P and Q lie on sides AB and AC respectively such that PQ is parallel to BC. Let D be the foot of the perpendicular from A to BC. Let M be the midpoint of PQ. Suppose that line segment DM meets the circumcircle of triangle APQ at a point X inside triangle ABC.

   Prove that $\angle AXB = \angle AXC$.

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DAY 2
Wednesday, 8 February 2023
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. Some consecutive positive integers have been written on a whiteboard. Leigh circles some of them and underlines some of them so that one more number is circled than underlined. (Numbers can be both circled and underlined.) It turns out that, no matter how Leigh does this, the sum of the circled numbers is always greater than the sum of the underlined numbers.

   Show that there is at most one square number on the whiteboard.

6. Let $ABCD$ be a fixed parallelogram with AB parallel to DC, and AD parallel to BC. A point E, different from A and B, is chosen on the side AB. Let K be the centre of the circle through A, D and E. Let L be the centre of the circle through B, C and E.

   Prove that no matter where E is chosen, the length KL is always the same.

7. Let n be a positive integer. A positive integer k is called a *benefactor* of n if the positive divisors of k can be partitioned into two sets A and B such that n is equal to the sum of the elements of A minus the sum of the elements of B. Note that A or B may be empty, and that the sum of the elements of the empty set is 0.

   For example, 15 is a benefactor of 18 because $(1 + 5 + 15) - (3) = 18$.

   Show that every positive integer n has at least 2023 benefactors.

8. Sam is playing a game with 2023 cards labelled $1, 2, 3, \ldots, 2023$. The cards are shuffled and placed in a pile face down. On each turn, Sam thinks of a positive integer n and then looks at the number on the topmost card. If the number on the card is at least n, then Sam gains n points; otherwise Sam gains 0 points. Then the card is discarded. This process is repeated until there are no cards left in the pile.

   Find the largest integer P such that Sam can guarantee a total of at least P points from this game, no matter how the cards were originally shuffled.

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