1. Let n be a positive integer greater than 2. Mia is playing the following game. She writes the numbers $1, 2, 3, \ldots, n$ in some order on the sides of a regular n-sided polygon, one number per side. Then, on each vertex of the polygon, she writes the sum of the numbers on the two sides that meet at that vertex. Mia wins if the n numbers on the vertices can be written down in some order to form an arithmetic progression.

   For which n can Mia win this game?

   (In an *arithmetic progression*, there is a constant d such that each term is equal to the previous term plus d.)

Solution (Sally Tsang)

Answer: Mia can win if and only if n is odd.

Suppose Mia wins for a certain n. Let the n numbers on the n vertices be an arithmetic progression with initial term a and difference d. So a and d are positive integers, $a \geq 3$ and $d \geq 1$, and the sum of all n terms is twice the sum of the numbers on the vertices. So:

$$\frac{(a + a + (n-1)d)n}{2} = n(n+1)$$
$$2a + (n-1)d = 2n + 2$$

If $d \geq 2$, then LHS > RHS. Hence $d = 1$. Simplifying yields $a = \frac{n+3}{2}$. So n must be odd.

When n is odd, a winning arrangement is given by:

$$k,\ 2k-1,\ k-1,\ 2k-2,\ k-2,\ 2k-3,\ \ldots,\ k+1,\ 1$$

It is easy to check that the n numbers on the vertices are the numbers from $k+1$ to $3k-1$, which is an arithmetic progression. Another way to visualise this is to place 1 on any side, then skip a side and place 2, skip a side and place 3 and so forth.

2. A set of integers is *wanless* if the sum of its elements is 1 less than a multiple of 4.

   How many subsets of $\{1, 2, 3, \ldots, 2023\}$ are wanless?

Solution 1 (Norman Do)

Answer: 2^{2021}

We will show that there are 2^{2021} of these subsets whose sum is divisible by 4, 2^{2021} of these subsets whose sum one more than a multiple of 4, 2^{2021} of these subsets whose sum two more than a multiple of 4, and 2^{2021} of these subsets whose sum three more than a multiple of 4.

The idea is to take all 2^{2023} subsets of $\{1, 2, 3, \ldots, 2023\}$ and to split them into 2^{2021} groups of four subsets so that in any group, the sums are distinct modulo 4. To do this, we say that two subsets are in the same group if they are the same set after removing any occurrences of the numbers 1 or 2. Then every group is of the form

$$X \qquad X \cup \{1\} \qquad X \cup \{2\} \qquad X \cup \{1, 2\},$$

where X is an arbitrary (possibly empty) subset of $\{3, 4, 5, \ldots, 2023\}$. This completes the proof.

Solution 2 (Alan Offer)

Suppose that a set A has the property $\mathcal{P}$ that its subset sums are evenly distributed over the four residue classes modulo 4.

Now consider a set $B = A \cup \{b\}$ obtained by augmenting A with one additional element. The subsets of B are of two types: those with b and those without b. Those without b, being subsets of A, have their sums evenly distributed over the four residue classes modulo 4. Those with b likewise have their sums evenly distributed over the four residue classes as their sums are as for the subsets of A increased by the common value b. Taken all together, it follows that B also has property $\mathcal{P}$.

Let $A = \{1, 2\}$. The four subsets of A are $\emptyset, \{1\}, \{2\}$ and $\{1, 2\}$, and their respective sums reduced modulo 4 are 0, 1, 2, 3. Thus A has property $\mathcal{P}$. Adding one element at a time, it follows that $S = \{1, 2, 3, \ldots, 2023\}$ has property $\mathcal{P}$, too. It follows that the number of subsets of S whose sums are in the residue class of 3 modulo 4 is $|S|/4 = 2^{2023}/2^2 = 2^{2021}$.

3. Let $f(x)$ be a polynomial with real coefficients not all equal to zero.

   Prove that there exists another polynomial $g(x)$ with real coefficients such that the polynomial $f(x)g(x)$ has exactly 2023 more positive coefficients than negative coefficients.

Solution (Ian Wanless)

For any $z \in \mathbb{R}$, define

$$\operatorname{sgn}(z) = \begin{cases} +1 & \text{if } z > 0 \\ 0 & \text{if } z = 0 \\ -1 & \text{if } z < 0 \end{cases}$$

and define

$$s\Big(\sum_{i=0}^{n} c_i x^i\Big) = \sum_{i=0}^{n} \operatorname{sgn}(c_i).$$

We note that it suffices to find $h(x)$ such that $|s(h(x)f(x))| = 1$. Since then for any d greater than the degree of $h(x)f(x)$ we can take

$$\big(1 + x^d + x^{2d} + \cdots + x^{2022d}\big)h(x)f(x)$$

or its negative, and this will have the desired property.

Let $f(x) = \sum_{i=a}^{b} c_i x^i$ where $b \geq a$ and $0 \notin \{c_a, c_b\}$.

Now consider $h(x) = x^{b-a} - \lambda$ where $\lambda > 0$ and $\lambda \neq c_a/c_b$. Then

$$\begin{aligned} s(h(x)f(x)) &= s\Big(\sum_{i=a}^{b-1}(-\lambda)c_i x^i + (c_a - \lambda c_b)x^b + \sum_{i=a+1}^{b} c_i x^{i+b-a}\Big) \\ &= \sum_{i=a}^{b-1} \operatorname{sgn}(-\lambda c_i) + \operatorname{sgn}(c_a - \lambda c_b) + \sum_{i=a+1}^{b} \operatorname{sgn}(c_i) \\ &= \operatorname{sgn}(c_a - \lambda c_b) - \sum_{i=a}^{b-1} \operatorname{sgn}(c_i) + \sum_{i=a+1}^{b} \operatorname{sgn}(c_i) \\ &= \operatorname{sgn}(c_a - \lambda c_b) - \operatorname{sgn}(c_a) + \operatorname{sgn}(c_b). \end{aligned}$$

There are two cases. If $\operatorname{sgn}(c_a) = \operatorname{sgn}(c_b)$ then $|s(h(x)f(x))| = |\operatorname{sgn}(c_a - \lambda c_b)| = 1$ (noting that we chose $\lambda \neq c_a/c_b$). Alternatively, if $\operatorname{sgn}(c_a) = -\operatorname{sgn}(c_b)$ then $\operatorname{sgn}(c_a - \lambda c_b) = \operatorname{sgn}(c_a)$ so $|s(h(x)f(x)| = |\operatorname{sgn}(c_b)| = 1$. Either way we have achieved our goal.

4. Let ABC be an acute triangle. Points P and Q lie on sides AB and AC respectively such that PQ is parallel to BC. Let D be the foot of the perpendicular from A to BC. Let M be the midpoint of PQ. Suppose that line segment DM meets the circumcircle of triangle APQ at a point X inside triangle ABC.

   Prove that $\angle AXB = \angle AXC$.

Solution (Xiaoyu Chen)

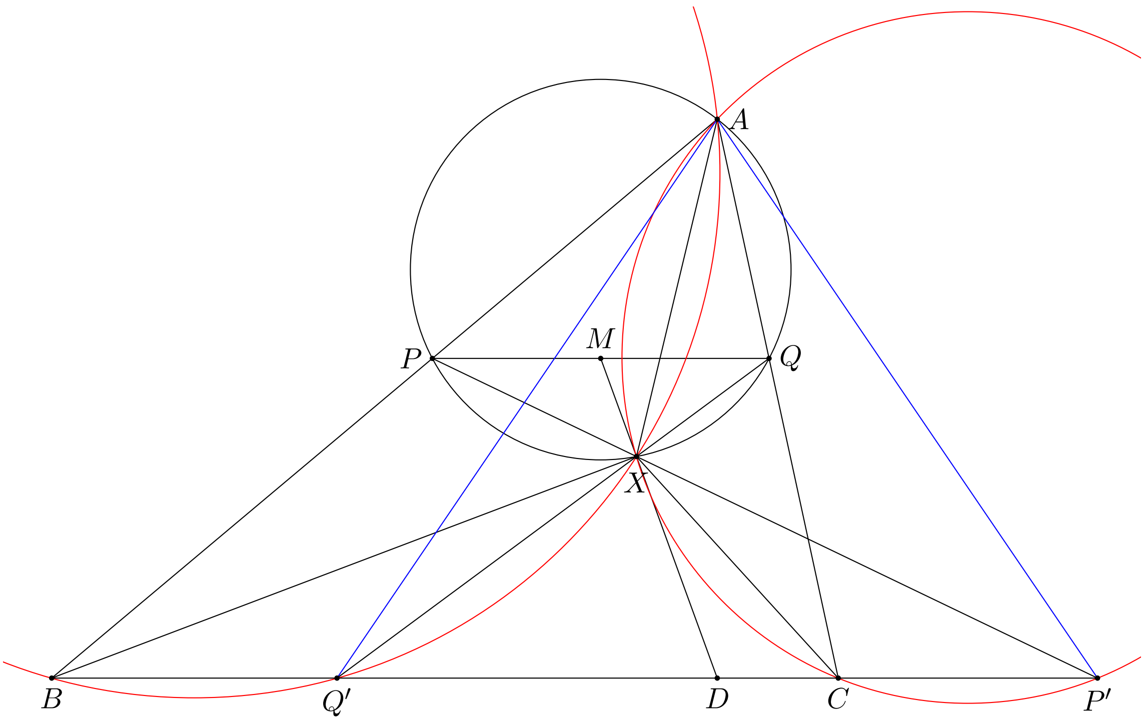
Assume all angles are directed counterclockwise (modulo 180°) in this solution where $\angle XYZ$ denotes $\angle(XY, YZ)$.

Let PX and QX meet BC at P' and Q'. Since $APXQ$ is cyclic and $PQ \parallel BC$, we have $\angle XAC = \angle XAQ = \angle XPQ = \angle XP'C$. Hence $AXP'C$ is cyclic. Similarly, $AXQ'B$ is cyclic.

By the dilation through X taking P, M, Q to P', D, Q', we have that $DP' = DQ'$. Combine this with $AD \perp P'Q'$, we see that $AP'Q'$ is isosceles and thus $AP' = AQ'$.

Therefore,

$$\begin{aligned}
\angle AXB &= \angle AQ'B \quad (AXQ'B \text{ cyclic}) \\
&= -\angle AP'C \ (AP'Q' \text{ isosceles}) \\
&= \angle CP'A \\
&= \angle CXA. \quad (AXP'C \text{ cyclic})
\end{aligned}$$



5. Some consecutive positive integers have been written on a whiteboard. Leigh circles some of them and underlines some of them so that one more number is circled than underlined. (Numbers can be both circled and underlined.) It turns out that, no matter how Leigh does this, the sum of the circled numbers is always greater than the sum of the underlined numbers.

   Show that there is at most one square number on the whiteboard.

Solution 1 (Angelo Di Pasquale)

Assume that, for the sake of contradiction, Leigh writes at least two square numbers. Since the numbers are consecutive, two of them are consecutive squares, say x^2 and $(x+1)^2$, where x is positive. By the given condition we have

$$\underbrace{x^2+(x^2+1)+\cdots+(x^2+x)}_{x+1 \text{ consecutive numbers}} > \underbrace{(x^2+x+1)+(x^2+x+2)+\cdots+(x^2+x+x)}_{x \text{ consecutive numbers}}$$

$$\begin{aligned} &\iff & (x+1)x^2 &> x(x^2+x) \quad (\text{cancel } 1+2+\cdots+x \text{ from both sides})\\ &\iff & x &> x+1 \end{aligned}$$

which is clearly false. Contradiction.

Solution 2 (Mikaela Gray)

Let the consecutive numbers be $a, a+1, \ldots, b$. Choose $k = \lfloor \frac{b-a}{2} \rfloor$ and suppose Leigh circles the $k+1$ smallest numbers and underlines the k numbers after that, we see that

$$a+(a+1)+\cdots+(a+k-1)+(a+k) > (a+k+1)+(a+2k-1)+\cdots+(a+2k).$$

Rearranging, this is equivalent to

$$a > k+k+\cdots+k \qquad \iff \qquad a > k^2.$$

Now if two squares x^2 and $(x+1)^2$ appear on the whiteboard, then $x^2 \geq a$. But

$$(x+1)^2 = x^2+2x+1 \geq a+2\sqrt{a}+1 > a+2k+1 \geq b,$$

which is a contradiction.

6. Let $ABCD$ be a fixed parallelogram with AB parallel to DC, and AD parallel to BC. A point E, different from A and B, is chosen on the side AB. Let K be the centre of the circle through A, D and E. Let L be the centre of the circle through B, C and E.

   Prove that no matter where E is chosen, the length KL is always the same.

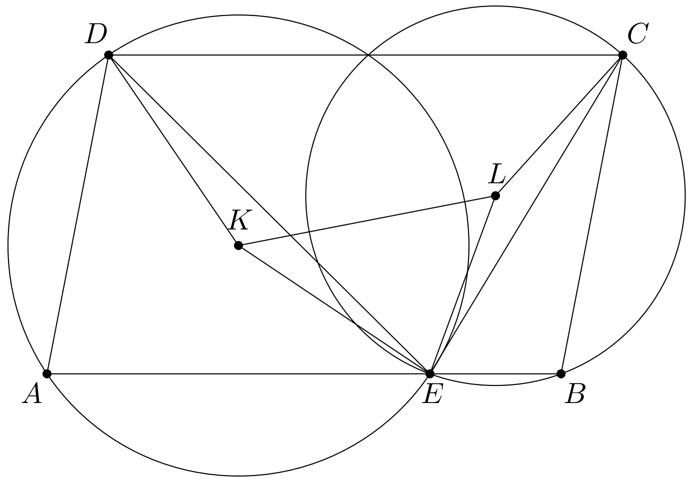
Solution 1 (Thanom Shaw)

From the problem, K is the point of intersection of the perpendicular bisectors of AD and AE, while L is the point of intersection of the perpendicular bisectors of BC and BE.

The perpendicular bisectors of AE and BE are both perpendicular to side AB and are a fixed distance ($\frac{1}{2}|AB|$) apart, with AE closer to A than BE. The perpendicular bisectors of AD and BC are both perpendicular to side AD and are independent of E. Hence these four perpendicular bisectors form a parallelogram, whose dimensions and orientation remain constant as E varies on AB.

Points K and L are opposite vertices of this parallelogram, as defined at the start of the solution. The length of KL, a (fixed) diagonal of the parallelogram, is independent of E.

Solution 2 (Angelo Di Pasquale)



Without loss of generality, let $\angle ABC \geq 90°$. Observe that

$$\angle ELC = 360° - \text{reflex}(\angle ELC) = 360° - 2\angle CBE = 2(180° - \angle CBE) = 2\angle EAD = \angle EKD.$$

Since $KE = KD$, and $LE = LC$, we have $\triangle EKD \sim \triangle ELC$ (PAP). Similar switching about E yields $\triangle EKL \sim \triangle EDC$. Thus

$$\frac{KL}{DC} = \frac{EK}{ED}. \tag{1}$$

But $\angle EKD = 2\angle EAD$ which does not depend on E. Hence the similarity type of $\triangle EKD$ does not depend on E. Thus RHS (1) does not depend on E. Since DC also does not depend on E, from (1) it follows that KL does not depend on E.

7. Let n be a positive integer. A positive integer k is called a *benefactor* of n if the positive divisors of k can be partitioned into two sets A and B such that n is equal to the sum of the elements of A minus the sum of the elements of B. Note that A or B may be empty, and that the sum of the elements of the empty set is 0.

   For example, 15 is a benefactor of 18 because $(1+5+15)-(3)=18$.

   Show that every positive integer n has at least 2023 benefactors.

Solution (Angelo Di Pasquale)

Let n be any integer. We shall show that there are infinitely many positive integers m such that $n=\sum_{i=1}^{k}\pm d_i$ where $d_1,d_2,\ldots,d_k$ are the positive divisors of m, and some suitably chosen combination of $\pm$ signs depending on m and n.

Case 1. n is odd.

Without loss of generality, assume $n>0$. Consider the binary representation $n=2^{a_1}+2^{a_2}+\cdots+2^{a_k}$, where $0=a_1<a_2<\ldots<a_k$ are non-negative integers. For each integer $r\geq a_k$, we turn this into a signed sum of the divisors of $m=2^r$ inductively as follows. Whenever 2^j $(j<r)$ is present in the signed sum for n but 2^{j+1} is not, then replace 2^j by $2^{j+1}-2^j$. Doing this finitely many times writes n as a signed sum of the divisors of $m=2^r$, as desired.

Case 2. n is even.

Since $n-3$ is odd, we may use case 1, to find $n-3$ as the signed sum of the positive divisors of 2^r for each integer r satisfying $2^r\geq n-3$. Consider $m=3\cdot 2^r$. Since $3=3(2^r-2^{r-1}-2^{r-2}-\cdots-1)$, we may write 3 as the signed sum of all divisors of $3\cdot 2^r$ that are multiples of 3. Adding together the two signed sums for $n-3$ and 3 writes n as the signed sum of all divisors of $m=3\cdot 2^r$, as desired.

(A variation: instead of splitting n into 3 and $n-3$, we can split it as $n=3a+b$ for odd a,b. The same argument can be used as long as $2^r\geq\max(a,b)$.)

8. Sam is playing a game with 2023 cards labelled $1, 2, 3, \ldots, 2023$. The cards are shuffled and placed in a pile face down. On each turn, Sam thinks of a positive integer n and then looks at the number on the topmost card. If the number on the card is at least n, then Sam gains n points; otherwise Sam gains 0 points. Then the card is discarded. This process is repeated until there are no cards left in the pile.

   Find the largest integer P such that Sam can guarantee a total of at least P points from this game, no matter how the cards were originally shuffled.

Solution 1 (Angelo Di Pasquale)

Answer: 1012^2

First note that 1012^2 is achievable if Sam picks 1012 every single time.

We shall show that Sam cannot do better than this if the deck responds as follows:

> On each turn, if Sam chooses the number c, then
> (i) If all remaining cards in the deck are at least c, then the deck responds with the largest number in the deck. (We shall colour each such card blue.)
> (ii) If there are any cards still left in the deck that are less than c, then the deck responds with the smallest number left in the deck. (Such cards are left uncoloured.)

From (i), the blue cards are turned over in the order $2023, 2022, \ldots, b$ for some $b \geq 1$.

Consider a number c that Sam chooses for which the deck responds with a blue card. Since the card is blue, (i) applies. Since (i) applies, the card numbered b is still in the deck. It follows that $c \leq b$. Since Sam gets 0 points for each uncoloured card, and at most b points for each blue card, Sam's total is at most

$$b(2023 - b + 1) = 1012^2 - (b - 1012)^2 \leq 1012^2$$

as required.

Solution 2 (Ivan Guo)

The largest point total that can be guaranteed is 1012^2. We first prove the following claim.

> Suppose there are m cards labelled with positive integers $a_1, a_2, \ldots, a_m$, such that $a_1 > a_2 > \cdots > a_m$, then the maximum point total that can be guaranteed with these cards is $\max_{i=1,\ldots,m} ia_i$.

First, we show that Sam can achieve $\max_{i=1,\ldots,m} ia_i = ka_k$, where k is the index that achieves the maximum. This can be done by thinking of the integer a_k at every turn. As there are exactly k cards with value a_k or greater, Sam would score a_k on exactly k turns and score 0 on the other turns.

We now prove that Sam cannot do better than $\max_{i=1,\ldots,m} ia_i$ via induction on the number of cards m. The result is certainly true for $m = 1$ since the maximum possible score is simply a_1. Assume this is true for m cards and consider the case of $m + 1$ cards, labelled $a_1, a_2, \ldots, a_{m+1}$ in descending order. Suppose that Sam thinks of the number n in the first turn. There are two cases.

- If $n > a_{m+1}$, then consider the scenario where the card a_{m+1} appears in the first draw. Sam would score 0 points in the first turn and is left with the cards $a_1, \ldots, a_m$. By the inductive hypothesis, at most $\max_{i=1,\ldots,m} ia_i$ points can be guaranteed in the remaining turns. Then the maximal point total is

$$0 + \max_{i=1,\ldots,m} ia_i \le \max_{i=1,\ldots,m+1} ia_i.$$

- If $n \le a_{m+1}$, then consider the scenario where the card a_1 appears in the first draw. Sam would score n points in the first turn and is left with the cards $a_2, \ldots, a_{m+1}$. By the inductive hypothesis, at most $\max_{i=2,\ldots,m+1}(i-1)a_i$ points can be guaranteed in the remaining turns. Then the maximal point total is

$$\begin{aligned} n + \max_{i=2,\ldots,m+1}(i-1)a_i &\le a_{m+1} + \max_{i=2,\ldots,m+1}(i-1)a_i \\ &\le \max_{i=2,\ldots,m+1} ia_i \\ &\le \max_{i=1,\ldots,m+1} ia_i. \end{aligned}$$

So the induction is complete.

Applying the claim to the problem, by the AM-GM inequality, the maximum point total Sam can guarantee is

$$\max_{i=1,\ldots,m} i(2024-i) = 1012^2,$$

which can be achieved by thinking of the number 1012 at every turn.

Statistics

Score Distribution/Problem

Mark/Problem	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8
0	10	47	97	151	2	47	51	53
1	4	10	8	0	6	3	9	82
2	15	11	2	2	7	9	11	8
3	19	3	1	1	5	5	9	0
4	9	3	2	0	4	3	19	0
5	8	2	5	0	2	7	11	3
6	44	9	19	0	8	29	9	4
7	55	79	30	10	130	61	45	14
Average	**5.0**	**4.1**	**2.3**	**0.5**	**6.2**	**4.2**	**3.4**	**1.4**

The average score was 26.5.

Cuts for Gold, Silver and Bronze awards were 45, 38 and 26, respectively.