AMO. 2024 AUSTRALIAN MATHEMATICAL OLYMPIAD

Day 1
Tuesday 6 February 2024
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Determine all triples (k, m, n) of positive integers satisfying

$$k! + m! = k! \times n!\,.$$

(If n is a positive integer, then $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.)

2. Let $ABCD$ be a cyclic quadrilateral. Point P is on line CB such that $CP = CA$ and B lies between C and P. Point Q is on line CD such that $CQ = CA$ and D lies between C and Q.

   Prove that the incentre of triangle ABD lies on line PQ.

   (The *incentre* of a triangle is the point where its angle bisectors intersect.)

3. Let $a_1, a_2, \ldots, a_n$ be positive real numbers, where $n \geq 2$. For each permutation $(b_1, b_2, \ldots, b_n)$ of $(a_1, a_2, \ldots, a_n)$, define its *score* to be

$$\frac{b_1^2}{b_2} + \frac{b_2^2}{b_3} + \cdots + \frac{b_{n-1}^2}{b_n}.$$

   Show that there exist two permutations of $(a_1, a_2, \ldots, a_n)$ whose scores differ by at least $3|a_1 - a_n|$.

4. Consider a 2024×2024 grid of unit squares. Two distinct unit squares are *adjacent* if they share a common side. Each unit square is to be coloured either black or white. Such a colouring is called *evenish* if every unit square in the grid is adjacent to an even number of black unit squares.

   Determine the number of evenish colourings.

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Day 2
Wednesday 7 February 2024
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. The sequence of positive integers $a_1, a_2, \ldots, a_{2025}$ is defined as follows:

   - $a_1 = 2^{2024} + 1$
   - for each $n = 1, 2, \ldots, 2024$, define a_{n+1} to be the largest prime divisor of $a_n^2 - 1$.

   Determine the value of $a_{2024} + a_{2025}$.

6. In a school, there are 1000 students in each year level, from Year 1 to Year 12. The school has 12 000 lockers, numbered from 1 to 12 000. The school principal requests that each student is assigned their own locker, so that the following condition is satisfied:

   > For every pair of students in the same year level, the difference between their locker numbers must be divisible by their year-level number.

   Can the principal's request be satisfied?

7. Let $ABCD$ be a square and let P be a point on side AB. The point Q lies outside the square such that $\angle ABQ = \angle ADP$ and $\angle AQB = 90°$. The point R lies on the side BC such that $\angle BAR = \angle ADQ$.

   Prove that the lines AR, CQ and DP pass through a common point.

8. Let $r = 0.d_0d_1d_2\ldots$ be a real number (written in decimal form) where $d_0, d_1, d_2, \ldots$ is an infinite sequence of digits.

   For each integer $n \geq 0$, let

   $$e_n = 10^n d_n + 10^{n-1} d_{n-1} + \cdots + 10 d_1 + d_0$$

   be the number formed by writing the digits $d_n, d_{n-1}, \ldots, d_1, d_0$ in order from left to right. (Leading zeros are permitted.)

   Suppose that $d_0 = 6$ and, for each integer $n \geq 0$, the number e_n is equal to the number formed by the rightmost $n+1$ digits of e_n^2.

   Prove that r is irrational.

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