

1. Determine all triples (k, m, n) of positive integers satisfying

$$k! + m! = k! \times n!.$$

(If n is a positive integer, then $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.)

Solution

Answer: $(r, r, 2)$ for $r \geq 1$, and $(r! - 2, r! - 1, r)$ for $r \geq 3$.

The equation is equivalent to

$$m! = k!(n! - 1).$$

If $n = 1$, there are no solutions.

If $n = 2$, we get $m = k$ which yields the family $(r, r, 2)$.

If $n \geq 3$, then $m > k$.

- If $m = k + 1$, we get $k + 1 = n! - 1$ which yields the family $(r! - 2, r! - 1, r)$.
- If $m \geq k + 2$, we get $(k + 1)(k + 2) \cdots m = n! - 1$ which is a contradiction because LHS is even and RHS is odd.

So we have the two solution families: $(r, r, 2)$ for $r \geq 1$ and $(r! - 2, r! - 1, r)$ for $r \geq 3$.

2. Let $ABCD$ be a cyclic quadrilateral. Point P is on line CB such that $CP = CA$ and B lies between C and P . Point Q is on line CD such that $CQ = CA$ and D lies between C and Q .

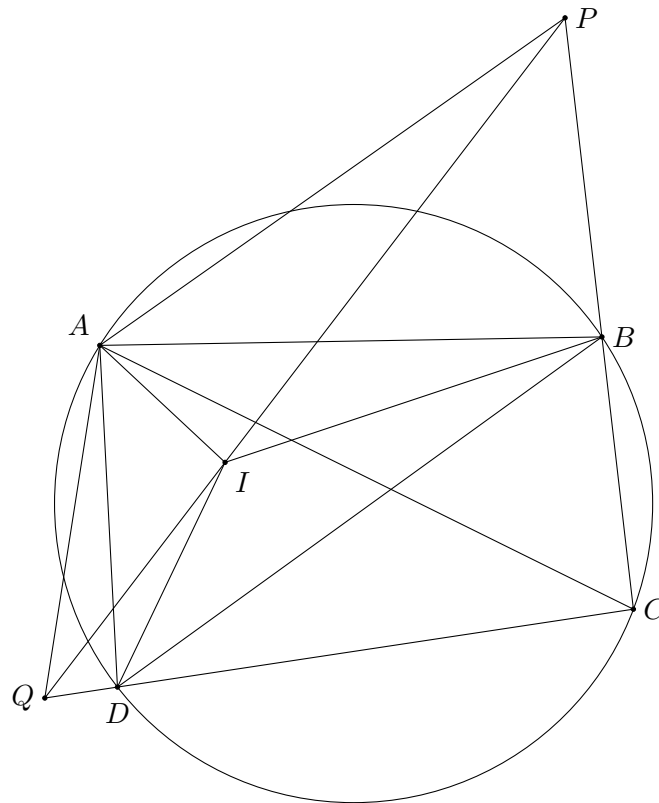
Prove that the incentre of triangle ABD lies on line PQ .

(The *incentre* of a triangle is the point where its angle bisectors intersect.)

Solution 1

Let I be the incentre of $\triangle ABD$. It suffices to show that $APBI$ and $AIDQ$ are cyclic, as that would imply the required collinearity via

$$\angle QIA + \angle AIP = \angle QDA + \angle ABP = 180^\circ.$$



Since $\triangle PCA$ is isosceles and $ABCD$ is cyclic,

$$\angle APC = 90^\circ - \frac{1}{2}\angle ACP = 90^\circ - \frac{1}{2}\angle ADB.$$

Using the fact that I is the incentre of $\triangle ABD$,

$$\angle AIB = 180^\circ - \frac{1}{2}(\angle DAB + \angle DBA) = 90^\circ + \frac{1}{2}\angle ADB.$$

Hence $\angle APC + \angle AIB = 180^\circ$ and $APBI$ is cyclic. By similar arguments, $AIDQ$ is also cyclic and the proof is complete.

Solution 2

Let the angle bisector of $\angle DAB$ meet PQ at I . It suffices to prove that IB is the angle bisector of $\angle ABD$.

First, since CPQ is isosceles and $ABCD$ is cyclic,

$$\angle IAB = \frac{1}{2}\angle DAB = 90^\circ - \frac{1}{2}\angle BCD = \angle IPC,$$

Hence $AIBP$ is cyclic. Then, since C is the centre of the circle through Q, A and P ,

$$\angle ABI = \angle APQ = \frac{1}{2}\angle ACQ = \frac{1}{2}\angle ABD,$$

IB is indeed the angle bisector of $\angle ABD$, as required.

Remark. There's a another variation of this solution which defines I to be the intersection of the angle bisector of $\angle ADB$ and PQ . Angle chasing yields $AIDQ$ is cyclic and hence I is the required incentre.

Solution 3

Let circles APB and AQD meet again at J . By (the converse of) the pivot theorem applied to $\triangle CPQ$ and circles $AQD, APB, CBDA$, the point J lies on line PQ .

Using circles $AJBP, PAQ$ and $ABCD$, we have

$$2\angle ABJ = 2\angle APJ = 2\angle APQ = \angle ACQ = \angle ACD = \angle ABD.$$

So BJ bisects $\angle ABD$. Similarly DJ bisects $\angle BDA$. So J is the incentre of $\triangle ABD$.

3. Let a_1, a_2, \dots, a_n be positive real numbers, where $n \geq 2$. For each permutation (b_1, b_2, \dots, b_n) of (a_1, a_2, \dots, a_n) , define its *score* to be

$$\frac{b_1^2}{b_2} + \frac{b_2^2}{b_3} + \dots + \frac{b_{n-1}^2}{b_n}.$$

Show that there exist two permutations of (a_1, a_2, \dots, a_n) whose scores differ by at least $3|a_1 - a_n|$.

Solution 1

We compare the scores of the decreasing and increasing permutations. First, reorder the sequence to $b_1 \geq b_2 \geq b_3 \geq \dots \geq b_n$. Consider the scores

$$\frac{b_1^2}{b_2} + \frac{b_2^2}{b_3} + \dots + \frac{b_{n-1}^2}{b_n} \quad \text{and} \quad \frac{b_n^2}{b_{n-1}} + \frac{b_{n-1}^2}{b_{n-2}} + \dots + \frac{b_2^2}{b_1}.$$

Their difference is

$$\frac{b_1^2}{b_2} - \frac{b_2^2}{b_1} + \dots + \frac{b_{n-1}^2}{b_n} - \frac{b_n^2}{b_{n-1}} = \frac{(b_1 - b_2)(b_1^2 + b_1b_2 + b_2^2)}{b_1b_2} + \dots + \frac{(b_{n-1} - b_n)(b_{n-1}^2 + b_{n-1}b_n + b_n^2)}{b_{n-1}b_n}.$$

Since for positive reals x, y , we have $x^2 + xy + y^2 \geq 3xy$ (which follows from $(x - y)^2 \geq 0$). Hence the above difference is at least

$$3(b_1 - b_2) + 3(b_2 - b_3) + \dots + 3(b_{n-1} - b_n) = 3(b_1 - b_n) \geq 3|a_1 - a_n|.$$

Solution 2

We may suppose without loss of generality that $a_1 \geq a_2 \geq \dots \geq a_n$, since this maximises the value of $3|a_1 - a_n|$ and we are considering scores over all permutations.

For $n = 2$, take the permutations (a_1, a_2) and (a_2, a_1) . The difference between the scores is then

$$\frac{a_1^2}{a_2} - \frac{a_2^2}{a_1} = \frac{a_1^3 - a_2^3}{a_1a_2} = (a_1 - a_2) \left(\frac{a_1}{a_2} + 1 + \frac{a_2}{a_1} \right) \geq 3(a_1 - a_2),$$

where the inequality follows from the AM-GM inequality.

For $n \geq 3$, take the permutations $(a_n, a_{n-1}, \dots, a_2, a_1)$ and $(a_1, a_{n-1}, \dots, a_2, a_n)$, which differ only in the first and last terms. Then the difference between the scores for these permutations is

$$\begin{aligned} \frac{a_1^2}{a_{n-1}} + \frac{a_2^2}{a_n} - \frac{a_n^2}{a_{n-1}} - \frac{a_2^2}{a_1} &= \frac{a_1^2 - a_n^2}{a_{n-1}} + a_2^2 \left(\frac{1}{a_n} - \frac{1}{a_1} \right) \\ &= (a_1 - a_n) \left(\frac{a_1}{a_{n-1}} + \frac{a_n}{a_{n-1}} + \frac{a_2^2}{a_1a_n} \right) \\ &\geq 3(a_1 - a_n), \end{aligned}$$

where the last inequality follows from the AM-GM inequality and $a_2 \geq a_{n-1}$. This completes the solution.

4. Consider a 2024×2024 grid of unit squares. Two distinct unit squares are *adjacent* if they share a common side. Each unit square is to be coloured either black or white. Such a colouring is called *evenish* if every unit square in the grid is adjacent to an even number of black unit squares.

Determine the number of evenish colourings.

Solution 1

Answer: 2^{2024}

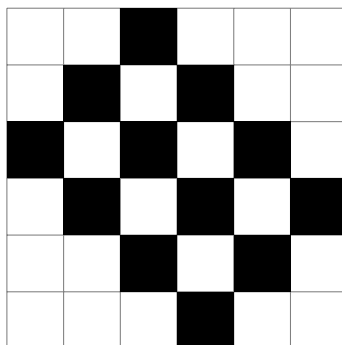
We will prove that the answer for the $m \times m$ grid case is 2^m .

Given two colourings A and B , we define their “sum” $A \oplus B$, a new colouring, in the following way. In the colouring $A \oplus B$, a square is coloured white if and only if it has the same colour in both A and B . This can also be viewed as a sum in modulo 2. Furthermore, this sum can be computed for more than two colourings, such as $A \oplus B \oplus C$ and so on.

Next, given two evenish colourings A and B , their sum $A \oplus B$ is also evenish. This is because, for a fixed square S , the number of black adjacent squares in $A \oplus B$ corresponds to the number of parity disagreements between A and B (adjacent to S), which must be even.

Fix the top row of a colouring. In order for the whole colouring to be evenish, the colouring of the second row is automatically determined, as every square in the top row must be adjacent to an even number of black squares. Continuing in this fashion, row by row, the colouring of the whole grid is automatically determined. The whole colouring is evenish as long as every square in the bottom row is adjacent to an even number of black squares. Thus the number of evenish colourings is at most the number of colourings for the top row, which is 2^m .

It remains to show that every colouring of the top row will determine an evenish colouring. We first construct the case where precisely one square in the top row is coloured black. This can be done using the following diamond-like pattern.



Note that this construction is symmetric under a rotation of 180° , so the adjacency condition for bottom row is automatically satisfied, just like the top row.

Finally, consider an arbitrary top row colouring C . It can be written as the sum of top row colourings C_1, C_2, \dots , each with only one black square. From before, each C_i determines a unique evenish colouring. By summing these evenish colourings, we obtain an evenish colouring with C as the top row. Therefore, every top row colouring determines a unique evenish colouring, and the number of evenish colourings must be 2^m .

Remark. Interested readers are invited to investigate the case of an $m \times n$ grid, which is much more difficult!

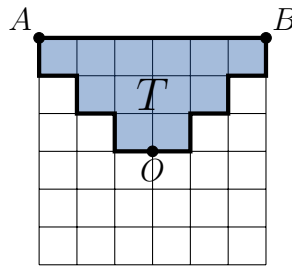
Solution 2

Call a unit square *good* if the number of black unit squares adjacent to it is even.

Note that the colours in the top row determine the numbers in the rest of the array row by row. The only issue is that this doesn't guarantee that the unit squares in the bottom row are necessarily good. We will show that they are indeed good.

Let T denote the triangular-like region consisting of all squares that overlap the interior of triangle ABO where A , B and O are the top-left vertex, top-right vertex and centre of the 2024×2024 grid. The colours of the squares in T are induced by the top row of the grid. Reflect T successively in lines AO, BO, AO, BO to fill in the colours of the squares in the rest of the grid while also returning T to exactly where it was at the outset.

All squares not on the main diagonals of the grid are good because they were good in the original T . And all squares on the main diagonals are also good due to symmetry.



5. The sequence of positive integers $a_1, a_2, \dots, a_{2025}$ is defined as follows:

- $a_1 = 2^{2024} + 1$
- for each $n = 1, 2, \dots, 2024$, define a_{n+1} to be the largest prime divisor of $a_n^2 - 1$.

Determine the value of $a_{2024} + a_{2025}$.

Solution We show that $\{a_{2024}, a_{2025}\} = \{2, 3\}$ and thus the answer is 5.

First observe that if $a_k = 2$ for some k , then from that point on the sequence values alternate between 2 and 3. We will show that a_{2024} is equal to either 2 or 3.

If for some $n \leq 2024$, a_n is even and thus equal to 2, then we are done. Thus we may suppose a_n is odd (and greater than 3) for $n \leq 2024$.

As a_n is odd, $a_n^2 - 1 = (a_n - 1)(a_n + 1) = 2 \times 2 \times \frac{(a_n - 1)}{2} \times \frac{(a_n + 1)}{2}$, and a_{n+1} is the maximum of 2, the largest prime divisor of $\frac{(a_n - 1)}{2}$ and the largest prime divisor of $\frac{(a_n + 1)}{2}$. In particular,

$$a_{n+1} \leq \frac{a_n + 1}{2}.$$

By repeated use of the above inequality we obtain $a_2 \leq 2^{2023} + 1$, $a_3 \leq 2^{2022} + 1$ and so on until we reach $a_{2024} \leq 2^1 + 1 = 3$.

Hence the value 2 or 3 has been reached at some point in the sequence no later than a_{2024} , forcing the sum of a_{2024} and a_{2025} to equal 5.

6. In a school, there are 1000 students in each year level, from Year 1 to Year 12. The school has 12 000 lockers, numbered from 1 to 12 000. The school principal requests that each student is assigned their own locker, so that the following condition is satisfied:

For every pair of students in the same year level, the difference between their locker numbers must be divisible by their year-level number.

Can the principal's request be satisfied?

Solution

The request can be satisfied.

First, the students in Year 12 are assigned the lockers numbered 12, 24, 36, \dots , 12 000 and the request is clearly satisfied for these students.

There are now 11 000 unassigned lockers remaining. Thus, by the pigeonhole principle, there are at least 1000 lockers whose numbers are all in the same equivalence class modulo 11. The Year 11 students are assigned these lockers. Because all these locker numbers have the same remainder modulo 11, the difference of any two is a multiple of 11, and the request is satisfied for the Year 11 students.

There are now 10 000 unassigned lockers remaining. Thus, there are at least 1000 lockers whose numbers are all in the same equivalence class modulo 10. The Year 10 students are assigned these lockers.

The process continues until finally there are precisely 1000 unassigned lockers remaining. The Year 1 students are assigned these lockers. The request is now satisfied for all students.

7. Let $ABCD$ be a square and let P be a point on side AB . The point Q lies outside the square such that $\angle ABQ = \angle ADP$ and $\angle AQB = 90^\circ$. The point R lies on the side BC such that $\angle BAR = \angle ADQ$.

Prove that the lines AR , CQ and DP pass through a common point.

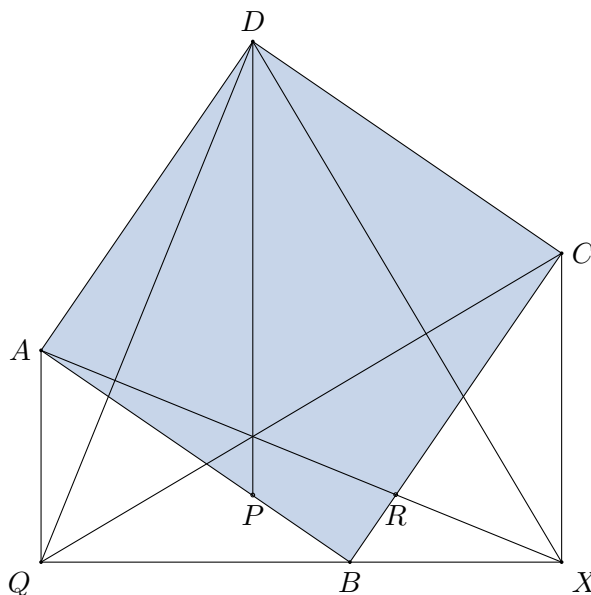
Solution 1

We know $\angle ADP = \angle ABQ \iff \angle DPA = \angle QAB$. Hence $DP \parallel AQ$, and so $DP \perp BQ$.

Let f denote the 90° rotation about the centre of the square that sends A to B . Let $X = f(Q)$. Thus $f(CDAQB) = DABXC$. Some consequences of this are:

- $\angle ADQ = \angle BAX$, which implies that A , R , and X are collinear.
- $\triangle AQB \cong \triangle BXC$ which implies that Q , B , and X are collinear.
- $DQ \perp AX$ and $QC \perp XD$

Thus the lines AX , QC , and DP are the altitudes of $\triangle DQX$, and are therefore concurrent. But these are the same as the lines AR , QC , and DP .



Solution 2

As in Solution 1, we have $DP \perp BQ$.

Let K denote the foot of the perpendicular from A to DP . Since $\angle ADK = \angle ABQ$, and $\angle DKA = 90^\circ = \angle BQA$, and $DA = BA$, it follows that $\triangle ADK \cong \triangle ABQ$ (AAS).

Let X be the point such that $\triangle ABQ \cong \triangle BCX$. An obvious angle consequence is that Q , B , and X are collinear. Another consequence is that $\triangle DAQ \cong \triangle ABX$ (SAS) since $DA = AB$, $AQ = BX$, and $\angle DAQ = \angle ABX$. Hence $\angle ADQ = \angle BAX$, which implies that A , R , and X are collinear.

Let $L = AX \cap QC$. We are required to show that DK passes through L . Let H be the foot of the perpendicular from L to BQ . It suffices to show $AK = QH$.

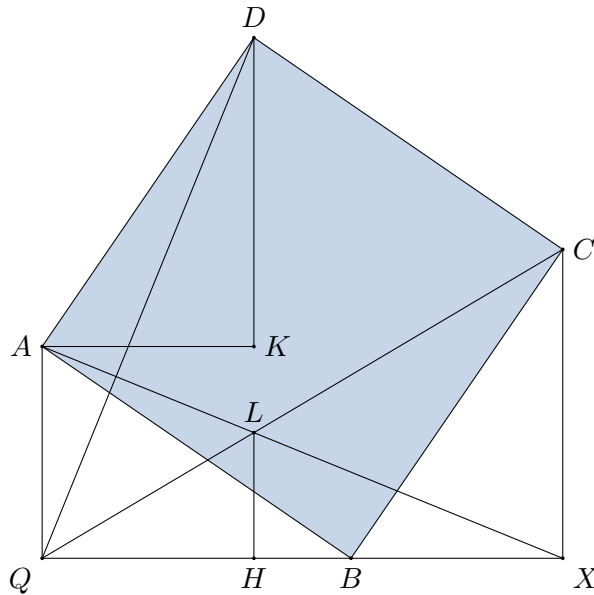
From $\triangle LAQ \sim \triangle LXC$ we have

$$\frac{XH}{HQ} = \frac{XL}{LA} = \frac{CX}{AQ} = \frac{BQ}{AQ}.$$

Adding 1 to both sides yields

$$\frac{BQ + AQ}{AQ} = \frac{XH + HQ}{HQ} = \frac{XQ}{HQ} = \frac{XB + BQ}{HQ} = \frac{AQ + BQ}{HQ}.$$

Thus $HQ = AQ = AK$, as desired.



8. Let $r = 0.d_0d_1d_2\dots$ be a real number (written in decimal form) where d_0, d_1, d_2, \dots is an infinite sequence of digits.

For each integer $n \geq 0$, let

$$e_n = 10^n d_n + 10^{n-1} d_{n-1} + \dots + 10d_1 + d_0$$

be the number formed by writing the digits $d_n, d_{n-1}, \dots, d_1, d_0$ in order from left to right. (Leading zeros are permitted.)

Suppose that $d_0 = 6$ and, for each integer $n \geq 0$, the number e_n is equal to the number formed by the rightmost $n + 1$ digits of e_n^2 .

Prove that r is irrational.

Solution 1

Since e_n^2 and e_n have the same rightmost $n + 1$ digits, their difference is a multiple of 10^n . Thus $10^n \mid e_n(e_n - 1)$. Since $e_n \equiv 6 \pmod{10}$ it follows that $e_n - 1$ is odd and e_n is not divisible by 5. Therefore we have

$$e_n \equiv 0 \pmod{2^n} \quad \text{and} \quad e_n \equiv 1 \pmod{5^n}. \quad (1)$$

Suppose for the sake of contradiction that $0.d_0d_1d_2\dots$ is rational so that d_0, d_1, d_2, \dots is periodic. Hence the sequence has a leading part with $r \geq 0$ terms, and then repeating cycle with $p \geq 1$ terms as follows.

$$\underbrace{d_0, \dots, d_{r-1}}_A, \underbrace{d_r, \dots, d_{r-1+p}}_B, \underbrace{d_{r+p}, \dots, d_{r-1+2p}}_B, \dots, \underbrace{d_{r+(k-1)p}, \dots, d_{r-1+kp}}_B, \dots \quad (2)$$

We will show that this is incompatible with (1).

Let $X = \overline{d_{r-1}d_{r-1}\dots d_0}$ and $Y = \overline{d_{r-1+p}d_{r-2+p}\dots d_r}$. For each positive integer k we consider e_{r-1+kp} . We have

$$\begin{aligned} e_{r-1+kp} &= X + 10^r Y + 10^{r+p} Y + \dots + 10^{r+(k-1)p} Y \\ &= X + Y \left(\frac{10^{r+kp} - 10^r}{10^p - 1} \right). \end{aligned}$$

Putting this into (1) yields

$$\begin{aligned} X + Y \left(\frac{10^{r+kp} - 10^r}{10^p - 1} \right) &\equiv 0 \pmod{2^{r-1+kp}} \\ \iff X(10^p - 1) - Y \cdot 10^r &\equiv 0 \pmod{2^{r-1+kp}} \end{aligned} \quad (3)$$

and

$$\begin{aligned} X + Y \left(\frac{10^{r+kp} - 10^r}{10^p - 1} \right) &\equiv 1 \pmod{5^{r-1+kp}} \\ \iff (X - 1)(10^p - 1) - Y \cdot 10^r &\equiv 0 \pmod{5^{r-1+kp}} \end{aligned} \quad (4)$$

Since the LHSs of (3) and (4) are constant and k can be arbitrarily large, this implies that

$$X(10^p - 1) - Y \cdot 10^r = 0 \quad \text{and} \quad (X - 1)(10^p - 1) - Y \cdot 10^r = 0.$$

Subtracting these last two equations yields $10^p - 1 = 0$, which is a contradiction as $p \geq 1$.

Solution 2

As in Solution 1, we deduce (1), or $e_n \equiv 0 \pmod{2^n}$ and $e_n \equiv 1 \pmod{5^n}$.

For the sake of contradiction, suppose $0.d_0d_1d_2\dots$ is rational, so there exist positive integers N and p such that $d_i = d_{i+p}$ for all $i > N$. Then for all $n > N$ we have

$$e_{n+p} - 10^p e_n = C$$

for some C independent of n .

Taking this equation mod 2^{n+p} , since (1) implies $e_{n+p} \equiv 0 \pmod{2^{n+p}}$ and $2^p e_n \equiv 0 \pmod{2^{n+p}}$, we find that

$$C = e_{n+p} - 5^p(2^p e_n) \equiv 0 \pmod{2^{n+p}}$$

for all large n , which implies that $C = 0$.

Working in mod 5^{n+p} , since (1) implies $e_{n+p} \equiv 1 \pmod{5^{n+p}}$ and $5^p e_n \equiv 5^p \pmod{5^{n+p}}$, we have

$$0 = C = e_{n+p} - 2^p(5^p e_n) \equiv 1 - 10^p \pmod{5^{n+p}}.$$

But this is a contradiction since $p > 0$. Therefore $0.d_0d_1d_2\dots$ is irrational.

Score Distribution/Problem

Mark/Problem	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8
0	7	84	93	100	22	150	141	184
1	6	15	29	24	6	3	6	12
2	20	13	14	16	24	0	3	0
3	16	3	3	9	17	0	1	0
4	8	2	4	9	17	0	2	1
5	25	2	0	10	10	1	4	1
6	59	2	8	3	19	1	1	1
7	60	80	50	30	86	46	43	2
Average	5.1	3.2	2.4	2	4.7	1.7	1.7	0.2

The average total score was 20.9 out of the maximum possible of 56.

Cuts for Gold, Silver and Bronze awards were 41, 30 and 17, respectively.¹

¹ AMO awards are given approximately as follows:

- Gold: top 10%.
- Silver: top 25%.
- Bronze: top 50%
- Honourable Mentions are awarded to those who get full marks for at least one problem, but who miss out on a Gold, Silver or Bronze award.