CHAPTER 14

Analysis

14.1 Real Numbers

Our first notion of the real numbers, is that they are the numbers that can be located somewhere on the **number line**, but apart from that, real numbers may be combined, two at a time, via operations $+$ and $\cdot$ and have certain properties. Mathematicians, of course, like to collect together commonalities, distil those elements which are somehow basic, in the sense, they cannot be proved — these give rise to *axioms* — and organise what's left in a sequence of *theorems* that discover the properties of what it is that is under scrutiny. In this way, often one can predict structure that wasn't apparent before. And so, mathematicians came to define an **abelian group** and build from that a **field**.

Definition 14.1.1. An **abelian group** $(G, *)$ is a set G with a binary operation $*$ satisfying:

- **G1:** $g, h \in G \implies g * h \in G$. (**closure**)
- **G2:** $\forall g, h, k \in G, (g * h) * k = g * (h * k)$. (**associativity**)
- **G3:** $\exists e \in G$ such that $\forall g \in G, e * g = g * e = g$. (identity)
  An element with the property $e * g = g * e = g$ for all $g \in G$, is called an **identity** element of G (under $*$).
  One can prove that when such an element e exists, then it is **unique**. So, let us write id_* for the unique element $e \in G$ satisfying $e * g = g * e = g$ for all $g \in G$.
- **G4:** $\forall g \in G, \exists h \in G$ such that $g * h = h * g = \mathrm{id}_*$. (inverse)
  Here id_* is the identity element determined in G3.
  For $g \in G$, an element $h \in G$ s.t. $g * h = h * g = \mathrm{id}_*$ is called an **inverse** of g (in G, under $*$). In fact, when such an element h exists, one can show it is unique, and so it is *the* inverse of g (in G, under $*$), and we write g^{-1} for the unique element h s.t. $g * h = h * g = \mathrm{id}_*$.
- **G5:** $\forall g, h \in G,\ g * h = h * g$. (**commutativity**)

The statements $\mathrm{G1}, \dots, \mathrm{G5}$ are called the **abelian group axioms**.

When the operation $*$ is $+$ (addition), instead of id_+ we write 0 (**zero**), and when the operation $*$ is $\cdot$ (multiplication), instead of $\mathrm{id}_\cdot$ we write 1 (**one**).

Also, when the operation $*$ is $+$, the unique *additive inverse* of g is written as $-g$ (rather than g^{-1}, which would be confusing!), and while less confusing we'll usually use the *reciprocal* notation for the *multiplicative inverse* of g, i.e. $\dfrac{1}{g}$, when it exists.

Definition 14.1.2. A **field** $(F, +, \cdot)$ is a set F with binary operations $+$ and $\cdot$ satisfying:

- **F1:** $(F, +)$ is an abelian group with identity 0,
- **F2:** $(F \setminus \{0\}, \cdot)$ is an abelian group with identity 1, and $(F, \cdot)$ only fails to be an abelian group, in that 0 has no inverse, and
- **F3:** $\forall x, y, z \in F,\ x \cdot (y + z) = x \cdot y + x \cdot z$. (**distributive law**)

Usually we write F instead of $(F, +, \cdot)$, where $+$ usually corresponds to ordinary *addition,* and $\cdot$ to ordinary *multiplication,* and we usually write xy rather than $x \cdot y$.

Remark 14.1.3. Implicit in axiom F2, is that $1 \neq 0$.

The Real Numbers $\mathbb{R}$ is a *field*.

It would be very boring if after developing such a theory of fields that there was just one example. In fact, the Rational Numbers $\mathbb{Q}$ and the Complex Numbers $\mathbb{C}$ are also fields, and there an infinite number of fields between $\mathbb{Q}$ and $\mathbb{C}$. All of these fields have an infinite number of elements, but aside from these there are also *finite fields*.

Let's write out all the rules we have identified for $\mathbb{R}$ again in expanded form, remembering that Rules 1–5 are F1 expanded, Rules 6–10 are F2 expanded and Rule 11 is F3.

14.2 Real Number Laws

For all $x, y, z \in \mathbb{R}$:

- **R1.** $x + y \in \mathbb{R}$. (closure under $+$)
- **R2.** $(x + y) + z = x + (y + z)$. (associativity under $+$)
- **R3.** $0 + x = x + 0 = x$ and $0 \in \mathbb{R}$. (additive identity)
- **R4.** There is an element $-x \in \mathbb{R}$ s.t. $x + (-x) = (-x) + x = 0$. (additive inverses)
- **R5.** $x + y = y + x$. (commutativity under $+$)
- **R6.** $x \cdot y \in \mathbb{R}$. (closure under $\cdot$)
- **R7.** $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. (associativity under $\cdot$)
- **R8.** $1 \cdot x = x \cdot 1 = x$ and $1 \in \mathbb{R}$. (multiplicative identity)
- **R9.** If $x \neq 0$, there is an element $\frac{1}{x} \in \mathbb{R}$ s.t. $x \cdot \frac{1}{x} = \frac{1}{x} \cdot x = 1$. (multiplicative inverses)
- **R10.** $x \cdot y = y \cdot x$. (commutativity under $\cdot$)
- **R11.** $x \cdot (y + z) = x \cdot y + x \cdot z$. (distribution)

These field laws are not the end of the story. Another property of $\mathbb{R}$ is that it is an *ordered* field. Note that $\mathbb{C}$ is not an ordered field.

Definition 14.2.1. An **ordered field** is a field F with a binary relation $<$ satisfying:

- **OF1:** $\forall x \in F,\ x \not< x$. (**irreflexivity**)
- **OF2:** $\forall x, y, z \in F,\ x < y$ and $y < z \implies x < z$. (**transitivity**)
- **OF3:** $\forall x, y \in F$, one of $x = y$, $x < y$, $y < x$ holds. (**weak trichotomy**)
- **OF4:** $\forall x, y, z \in F,\ x < y \implies x + z < y + z$. (**compatibility w.r.t. addition**)
- **OF5:** $\forall x, y, z \in F,\ x < y$ and $0 < z \implies xz < yz$. (**compatibility w.r.t. multiplication**)

In an ordered field F, we define $x > y$ to mean $y < x$. Similarly, $x \leqslant y$ means $x < y$ or $x = y$, and $x \geqslant y$ means $y \leqslant x$. An element $x \in F$ is called **positive** if $x > 0$, **negative** if $x < 0$, **nonnegative** if $x \geqslant 0$ and **nonpositive** if $x \leqslant 0$.

We express the 'compatibility with respect to addition' property by saying that adding any field element to an inequality **preserves** the inequality. Notice how the 'compatibility with respect to multiplication' property is a little different: multiplying an inequality by a *positive* field element **preserves** the inequality.

One can show that the axioms OF1–OF5 imply

- **OF3*:** $\forall x, y \in F$, *exactly* one of $x = y$, $x < y$, $y < x$ holds. (**strong trichotomy**) and that replacing OF1 and OF3 with OF3* one obtains an alternative Ordered Field axiom system.

> Consider the sequence
>
> $$3, 3.1, 3.14, 3.141, 3.1415, \ldots$$
>
> containing decimal approximations of π to successively more decimal places. The **limit** of the sequence is π, but every number in the sequence is **rational**. The above sequence is an example of a **Cauchy sequence** which has the property that its elements get arbitrarily close to one another. If a set of numbers has the property that all *Cauchy sequences* have a limit, it is said to be **complete**; $\mathbb{R}$ is *complete* but $\mathbb{Q}$ is not.

We have only mentioned the operations $+$ and $\cdot$ on $\mathbb{R}$ so far. What about $-$ and $/$? We define **subtraction** as the addition of an *additive inverse:*

$$x - y := x + (-y).$$

Similarly, we define **division** as the multiplication by a *multiplicative inverse:*

$$x/y := x \cdot \frac{1}{y}.$$

Every field can be shown to have the following properties.

14.3 Properties of a field F.

1. $\forall x \in F$, $x \cdot 0 = 0$.
2. If $x, y \in F$ and $x \cdot y = 0$ then either $x = 0$ or $y = 0$.

We finish this section with an example of the consequences of the ordering rules.

Lemma 14.3.1. *If x and y are* positive *then*

$$x < y \iff x^2 < y^2.$$

Proof. We assume $x, y > 0$.
($\Longrightarrow$) Suppose $x < y$. Then

$$\begin{aligned} x^2 &< xy, && \text{by OF5, using } x > 0 \text{ and } x < y && (14.3.1)\\ \text{and } xy &< y^2, && \text{by OF5, using } y > 0 \text{ and } x < y && (14.3.2)\\ x^2 &< y^2, && \text{by OF3, using (14.3.1) and (14.3.2)} \end{aligned}$$

($\Longleftarrow$) Now suppose $x^2 < y^2$. Then

$$\begin{aligned} x^2 - y^2 &< 0\\ (x-y)(x+y) &< 0\\ (x-y)(x+y)(x+y)^{-1} &< 0, && \text{since } x, y > 0 \implies x+y > 0 \text{, so that by Law 9., } (x+y)^{-1}\\ &&& \text{exists, and then } 0 \cdot (x+y)^{-1} = 0 \text{ by Property 1.}\\ x - y &< 0, && \text{using Laws 9. and 8. to simplify}\\ x &< y. \end{aligned}$$

□

14.4 Absolute Value

Definition 14.4.1. For $x \in \mathbb{R}$, the **absolute value** of x, written $|x|$, is defined by

$$|x| = \begin{cases} x, & \text{if } x \geqslant 0, \\ -x, & \text{if } x < 0. \end{cases}$$

From this definition, it follows that in general

$$|x - a| = \begin{cases} x - a, & \text{if } x \geqslant a, \\ -x + a, & \text{if } x < a, \end{cases}$$

for any $a \in \mathbb{R}$.

A useful observation is that $|b - a|$ is the **distance** between $a, b \in \mathbb{R}$ on the number line. Also, for $a > 0$,

$$\begin{aligned} |x| < a &\iff -a < x < a \iff (-a < x) \text{ and } (x < a) \\ |x| > a &\iff (x < -a) \text{ or } (x > a). \end{aligned}$$

Note, that without the condition $a > 0$, the above statements are somewhat vacuous, e.g. for $a < 0$, $|x| < a$ is equivalent to saying that x belongs to the empty set!

The *absolute value* function, satisfies the following properties:

$$|xy| = |x|\,|y|, \tag{14.4.1}$$
$$\big||x| - |y|\big| \leqslant |x + y| \leqslant |x| + |y|. \tag{14.4.2}$$

The inequality (14.4.2) is the extended version of the **Triangle Inequality**.

14.5 Subsets of the Real Numbers

Many subsets of $\mathbb{R}$ can be described as unions of *intervals*.

Definition 14.5.1. The *set* of all points x that satisfy the inequality $a < x < b$, for some constants $a, b \in \mathbb{R}$ is the **interval** between a and b which is represented by

$$(a, b).$$

The round brackets indicate that the endpoints a and b are *not* included, and (a, b) is said to be an **open** *interval.*

If the endpoints are included, i.e. we have $a \leqslant x \leqslant b$, then the *interval* is **closed** and we use square brackets:

$$[a, b].$$

An interval may also be half open and half closed (sometimes referred to as **clopen** intervals), e.g. the interval such that $a \leqslant x < b$, is represented as

$$[a, b).$$

In short, a square bracket indicates the point is included and a round bracket means it isn't.

Example 14.5.2. 1. $(1,3) = \{x \mid 1 < x < 3\}$ *is an* open interval.

2. $[2,4] = \{x \mid 2 \leqslant x \leqslant 4\}$ *is a* closed interval.

3. $(3,5] = \{x \mid 3 < x \leqslant 5\}$ *and* $[3,5) = \{x \mid 3 \leqslant x < 5\}$ *are each* half-open *(or* clopen*)* intervals.

4. $[2,\infty) = \{x \mid x \geqslant 2\}$, $(2,\infty) = \{x \mid x > 2\}$, $(-\infty,2] = \{x \mid x \leqslant 2\}$, $(-\infty,2) = \{x \mid x < 2\}$ *are* infinite intervals.

5. $\{x \mid |x| \leqslant 1\} = [-1,1]$.

6. $\{x \mid |x| > 1\} = (-\infty,-1) \cup (1,\infty)$.

7. $\bigcup_{n=1}^{\infty} \left(n, n+\frac{1}{n}\right)$ *is an* infinite union of intervals *of decreasing size.*

The set above is an example of an *open set,* a *set* of points with a "fuzzy" boundary (in the sense that all boundary points are missing). Adding in all missing boundary points gives a *closed set.* Infinite unions of *open sets* are again *open,* but infinite intersections of *open sets* need not be open. Similarly, infinite unions of *closed sets* need not be *closed,* but infinite intersections of *closed sets* are *closed.*

Note that ∞ *is* not *a* real number; *it represents something that is* larger *than any* real number. *Similarly,* $-\infty$ *is less than any* real number.

Infinite sets tend to have properties that seem paradoxical. The rationals $\mathbb{Q}$ is an infinite subset of $\mathbb{R}$ with the following properties:

(i) Between each pair of rationals there is an irrational.

(ii) Between each pair of irrationals there is a rational.

There are also different *infinities.* The size (*cardinality*) of $\mathbb{N}$ is what's called a *countable* infinity, denoted by $\aleph_0$ ("aleph null"). It turns out that the cardinality of each of $\mathbb{Z}$ and $\mathbb{Q}$ is also $\aleph_0$. This seems strange since $\mathbb{Z}$ has an infinite number of elements that are not in $\mathbb{N}$. Similarly, $\mathbb{Q}$ has an infinite number of elements that are not in $\mathbb{Z}$. On the other hand, $\mathbb{R}$ is a lot bigger than any of $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$. It is said to be *uncountable* and has cardinality denoted by $\aleph_1$ or c, and there are larger infinities! Things become somewhat counter-intuitive at the infinite level.

14.6 Functions

Definition 14.6.1. Given two sets X and Y, a **function** f is a rule that associates with each $x \in X$, an *unique* element $y \in Y$. In this case, we write $y = f(x)$, and say that y is the **value** or **image** of the function f at x. The set X is the **domain** of f; and the set Y is the **codomain** of f. In this case, we may write:

$$f : X \to Y$$

and say, f **maps** X **into** Y. The **image** of f, also called the **range** of f, is the set

$$\{f(x) \mid x \in X\}.$$

In general the *range* of f is a subset of Y. If the *range* of f equals Y, then we say that f **maps** X **onto** Y. We may write, $\operatorname{dom}(f)$ and $\operatorname{range}(f)$, for the *domain* of f and *range* of f, respectively. Also, for each $x \in X$ and $y \in Y$ such that $y = f(x)$, we say that f **maps** x **to** y, and we may write

$$\begin{aligned} f : X &\to Y \\ x &\mapsto y. \end{aligned}$$

Note the slightly different shape of the right arrow for individual points: we write $f : X \to Y$ (map between sets) and $f : x \mapsto y$ (map between points).

Note that an element x can only be in the *domain* of f, if f has a *value* at x, i.e. if $f(x)$ is **defined**.

Example 14.6.2. *Real-valued functions of real variables are functions whose domain and codomain are both (subsets of)* $\mathbb{R}$. *Here are some examples.*

1. $f : \mathbb{R} \to \mathbb{R}$ *such that* $f(x) = 2x + 1$.
2. $f : [1, \infty) \to (0, \infty)$ *such that* $f(x) = \sqrt{x - 1}$.
3. $f : \mathbb{R} \to \mathbb{Z}$ *such that* $f(x) = \lfloor x \rfloor$, *where* $\lfloor x \rfloor$ *is the* ***floor*** *(or "integer part" of* x*), the* largest *integer that is* $\leqslant x$.

There is also the **ceiling** $\lceil x \rceil$ of x, which is the *smallest* integer that is $\geqslant x$. In general, for $x \in \mathbb{R}$, we have

$$\begin{aligned} \lfloor x \rfloor = x = \lceil x \rceil\,, &\text{ if } x \in \mathbb{Z}, \text{ and} \\ \lfloor x \rfloor < x < \lceil x \rceil\,, &\text{ if } x \notin \mathbb{Z}. \end{aligned}$$

For a real function f, *its* ***graph*** $\Gamma(f)$ *is the set of ordered pairs* $\big(x, f(x)\big)$ *for all* x *for which* f *is defined. Thus a real function's graph is a subset of* $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$. *Hence, in summary, we have*

$$\Gamma(f) = \{\big(x, f(x)\big) \mid x \in \operatorname{dom}(f)\} \subseteq \mathbb{R}^2.$$

This algebraic meaning of the graph $\Gamma(f)$ *of* f, *doesn't stop us from identifying a geometric interpretation, e.g. the graph* $\Gamma(f)$ *of* $f(x) = x^2, x \in \mathbb{R}$ *is a* parabola.

If f and g are two functions, then $f + g$, $f - g$, $f \cdot g$ and f/g are the functions defined **pointwise** (i.e. for each point x) by

$$\begin{aligned} (f + g)(x) &= f(x) + g(x) \\ (f - g)(x) &= f(x) - g(x) \\ (f \cdot g)(x) &= f(x) \cdot g(x) \\ (f/g)(x) &= f(x)/g(x) \end{aligned}$$

on the domain $\operatorname{dom}(f) \cap \operatorname{dom}(g)$, except that $f(x)/g(x)$ is not defined whenever $g(x) = 0$, so that

$$\operatorname{dom}(f/g) = \big(\operatorname{dom}(f) \cap \operatorname{dom}(g)\big) \setminus \{x \mid g(x) = 0\}.$$

For two functions $g : X \to Y$ and $f : Y \to Z$, we can also define the **composition** $f \circ g$ of f and g with domain X by

$$(f \circ g)(x) = f\big(g(x)\big),$$

so that $f \circ g : X \to Z$.

More generally, if instead $f : U \to Z$ and U contains Y, the same definition for $f \circ g$ applies, but if $U \subset Y$, then we must find the largest **restriction** $g_{X'}$ defined on a subset X' of X such that $g_{X'} : X' \to U$. Then $f \circ g$ is defined to be the function $f \circ g_{X'} : X' \to Z$.

Note. When the domain of a real function f is not given explicitly, it is taken to be the the largest subset of $\mathbb{R}$ on which f can be defined.

Example 14.6.3. 1. *$h(x) = \sin(2x^2 + 1)$ is a composite of $f(x) = \sin x$ and $g(x) = 2x^2 + 1$, with $\mathrm{dom}(h) = \mathrm{dom}(g) = \mathbb{R}$ and $\mathrm{range}(h) = \mathrm{range}(f) = [-1, 1]$.*

Note that since $\mathrm{range}(g) \neq \mathrm{dom}(f)$, a little checking is necessary to see that $\mathrm{range}(h)$ is indeed all of $\mathrm{range}(f)$ here (observe that $[\pi/2, 3\pi/2] \subset \mathrm{range}(g)$ is enough to show this).

Note that once we have delved this far into Analysis, the trigonometric functions are defined with their arguments in *radians*. So we have $\sin(\pi/2) = 1$.

2. *Let $f(x) = \sqrt{x}$ and $g(x) = 1 - x^2$. Then $\mathrm{dom}(f) = [0, \infty)$ and $\mathrm{dom}(g) = \mathbb{R}$. So $\mathrm{dom}(g \circ f) = [0, \infty)$, despite the fact that*

$$(g \circ f)(x) = g\big(f(x)\big) = g(\sqrt{x}) = 1 - (\sqrt{x})^2 = 1 - x,$$

i.e. despite the fact that $h(x) = 1 - x$ can be defined with domain $\mathbb{R}$, we must respect where $g \circ f$ came from when determining its domain; its domain cannot be larger than $\mathrm{dom}(f)$.

On the other hand,

$$(f \circ g)(x) = f\big(g(x)\big) = f(1 - x^2) = \sqrt{1 - x^2},$$

is only defined for $1 - x^2 \geqslant 0$, i.e. for $-1 \leqslant x \leqslant 1$. So in this case, $\mathrm{dom}(g) = [-1, 1]$ a proper subset of $\mathrm{dom}(g) = \mathbb{R}$. This possibility was foreshadowed in the dangerous bend that appeared before these examples. Putting that information in another way, we have that:

$$f \circ g \text{ is defined at } x \iff \begin{cases} g \text{ is defined at } x, \text{ and} \\ f \text{ is defined at } g(x). \end{cases}$$

14.7 Inverse functions

Definition 14.7.1. A function $f : X \to Y$ is said to be **one-to-one** (or **injective**) if for each $y \in \mathrm{range}(f)$ there is only one element $x \in X$ such that $f(x) = y$. Equivalently, f is *one-to-one* if and only if

$$f(x_1) = f(x_2) \implies x_1 = x_2.$$

What we see here is a frequent strategy in mathematics: to prove *uniqueness* of something, take two of whatever it is that has the property and show that, in fact, they are the same.

In topology, for a function $f : X \to Y$ one defines

$$f^{-1}(y) = \{x \in X \mid f(x) = y\}.$$

So in the case, where y is not in the range of f, $f^{-1}(y)$ is the empty set, and in the case where there are many values of $x \in X$ such that $f(x) = y$, $f^{-1}(y)$ is a set containing many elements. To guarantee $f^{-1}(y) \neq \emptyset$, we need $\mathrm{range}(f) = Y$, i.e. f must be *onto*. If also, $f^{-1}(y)$ are always sets containing just one element, which is to say that f is *one-to-one*, then we may define a *function* from Y to X, by essentially stripping the curly braces from each singleton set $f^{-1}(y)$. It is customary, to use the same notation for such a function in Real Analysis. Thus we have the following definition.

Definition 14.7.2. For a function $f : X \to Y$ that is both *onto* and *one-to-one,* the **inverse function** $f^{-1} : Y \to X$ is defined by

$$f^{-1}(y) = x \iff f(x) = y.$$

A function that is *onto* is also said to be **surjective** (*sur* is *on* in French). A function that is both *onto* and *one-to-one* is thus both *surjective* and *injective* and so it won't surprise you too much that such a function is said to be *bijective,* and since we have just seen that these are precisely the functions that have *inverses,* such functions are also called *invertible* functions.

An **identity** function (on a set X) is a function $\mathrm{id} : X \to X$ such that $\mathrm{id}(x) = x$ for all $x \in X$, which is to say it returns what you give it unchanged.

Now, if $f : X \to Y$ is *invertible,* i.e. $f^{-1} : Y \to X$ exists, then $f^{-1} \circ f$ is the composite function that does

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{f^{-1}} & X \\ x & \mapsto & y & \longmapsto & x \end{array}$$

in one step, i.e. $f^{-1} \circ f : x \mapsto x$ for all $x \in X$, so that $f^{-1} \circ f$ is the *identity* function on X. Similarly, $f \circ f^{-1}$ is the composite function that does

$$\begin{array}{ccccc} Y & \xrightarrow{f^{-1}} & X & \xrightarrow{f} & Y \\ y & \longmapsto & x & \mapsto & y \end{array}$$

in one step, i.e. $f \circ f^{-1} : x \mapsto x$ for all $y \in Y$, so that $f \circ f^{-1}$ is the *identity* function on Y.

Conversely, if there is a function g such that $g \circ f : x \mapsto x$ for all $x \in X$ and $f \circ g : y \mapsto y$ for all $y \in Y$ then $g = f^{-1}$ and $f = g^{-1}$.

Example 14.7.3. 1. *Let $f : [0, \infty) \to [0, \infty)$ such that $f(x) = x^n$ where $n \in \mathbb{N}$. Then f has an inverse and $f^{-1} = x^{\frac{1}{n}}$.*

2. *Let $f : [0, \infty) \to (0, 1]$ such that*

$$f(x) = \frac{1}{x^2 + 1}.$$

Then f is invertible with $f^{-1}(y) = \sqrt{\frac{1}{y} - 1}$.

3. *If $f(x) = 3x - 2$ then $f^{-1}(y) = \frac{1}{3}(y + 2)$.*

To find a formula for f^{-1} when it exists, replace $f(x)$ by y and, if possible, isolate y. Generally, an explicit formula for f^{-1} can only be found if a composition of inverse functions be applied in order to isolate y.

If f and g are the inverses of one another then their graphs $\Gamma(f)$ and $\Gamma(g)$ are reflections of one another in the line $y = x$, e.g. sketch $f(x) = x^2, x \in [0, \infty)$ and $g(x) = \sqrt{x}$.

14.8 Indices and Logarithms

Suppose that $x = p/q \in \mathbb{Q}$, where $p, q \in \mathbb{Z}$ and $q > 0$, and $a > 0$. Then we may define a function $\exp_a$ by

$$\exp_a(x) := a^{\frac{p}{q}} = (a^p)^{1/q} = \sqrt[q]{a^p},$$

where for $b = a^p > 0$, $b^{1/q} = \sqrt[q]{b}$ is the *positive real* q^{th} *root* of b.

We can *extend* this definition to have $x \in \mathbb{R}$ by *continuity* — essentially this means for any sequence of rational numbers $p_1/q_1, p_2/q_2, \dots$, with each of $q_1, q_2, \dots > 0$, that approach a given $x \in \mathbb{R}$, the sequence $a^{p_1/q_1}, a^{p_2/q_2}, \dots$ approaches some real number y — a^x is then defined to be y.

Now, why do we need $a > 0$? ...Well, if $a = 0$ then $\exp_a(x)$ is undefined for nonpositive x; and we avoid negative values of a since otherwise we encounter conflicts like the following:

- $a^{\frac{1}{3}}$ exists if we interpret this as the real cube root of a (then the value of $a^{\frac{1}{3}}$ is *negative* for negative a);
- $a^{\frac{2}{6}}$ ought to be interpreted as $(a^2)^{\frac{1}{6}}$ which is the positive 6^{th} root of the positive number a^2 (i.e. $a^{\frac{2}{6}}$ would necessarily be *positive* for negative a); and
- $\frac{1}{3} = \frac{2}{6}$.

So, insisting that $a > 0$ ensures that $\exp_a$ is *well-defined*.

The **index laws** which hold whenever their component parts are defined, are:

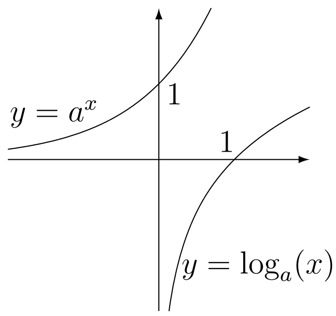
1. $a^x a^y = a^{x+y}$,
2. $(a^x)^y = a^{xy}$, and
3. $a^x b^x = (ab)^x$.

From these one may deduce the further rules:

4. $a^0 = 1$,
5. $a^1 = a$,
6. $a^{-x} = \dfrac{1}{a^x}$, and
7. $a^x/a^y = a^{x-y}$.

Now, the **logarithm** to **base** a of b, written $\log_a(b)$ is the power x that a must be raised to get b. For this definition to make sense we need $a > 0$ but $a \neq 1$ and $b > 0$. (Usually a is also an integer but this is not necessary.) More precisely then, for $1 \neq a > 0$ and $b > 0$,

$$\log_a(b) = x \iff a^x = b.$$



Note, that the $\log_a$ function is the inverse function of the $\exp_a$ function defined above.

For $0 < a < 1$, $\log_a$ is a *decreasing* function. If we let $c = \frac{1}{a}$, then $c > 1$ and

$$\begin{aligned}\log_a(b) = x &\iff a^x = b\\ &\iff c^{-x} = \left(\frac{1}{a}\right)^{-x} = b\\ &\iff \log_c(b) = -x\\ &\iff -\log_c(b) = x\end{aligned}$$

i.e. $\log_a$ and $-\log_c$ are the same function, which is to say that we don't really get any new functions by considering $0 < a < 1$. So it is more usual to consider only the functions $\log_a$ for $a > 1$, which are all *increasing* functions. We needed to avoid $a = 1$, since

$$\exp_1(x) = 1^x = 1 \text{ for all } x,$$

i.e. $\exp_1$ is not *one-to-one* and hence is not invertible. Thus no function $\log_1$ can be defined.

What are commonly called the **log laws** follow from the first two *index laws* (by putting $b = a^x$ and $c = a^y$):

1. $\log_a(bc) = \log_a(b) + \log_a(c)$,
2. $\log_a(b^y) = y\log_a(b)$.

Properties equivalent to Rules 4–7. are similarly deduced:

$$\begin{aligned}\log_a(1) &= 0,\\ \log_a(a) &= 1,\\ \log_a\left(\frac{1}{b}\right) &= -\log_a(b),\\ \log_a\left(\frac{b}{c}\right) &= \log_a(b) - \log_a(c).\end{aligned}$$

Interpreting Index Law 2. differently, we obtain the **change of base rule**:

$$\log_b(c) = \frac{\log_a(c)}{\log_a(b)}.$$

Proof. We start with Index Law 2. and put $b = a^x$ and $b^y = c$. Then

$$\begin{aligned}(a^x)^y &= b^y = c = a^{xy}\\ \therefore \log_b(c) &= y\\ &= \frac{xy}{x}\\ &= \frac{\log_a(c)}{\log_a(b)}.\end{aligned}$$

□

With $c = a$, from the *change of base rule* it follows that:

$$\log_b(a) = \frac{1}{\log_a(b)}.$$

14.9 Limits

Limits are the essence of Analysis. Many properties that are "intuitively obvious" require careful application of limits to be proved formally. Newton and Leibniz knew they were on the right track when they defined *derivatives* in terms of their understanding of limits, but it wasn't until Cauchy came up with the formal definition of limit that all their notions could be proved to be correct.

Suppose f is a function defined on some open *neighbourhood* of $x = a$, i.e. f is defined at every point of an open interval containing the point $x = a$, except possibly a itself. Then we write $f(x) \to L$ as $x \to a$, or, equivalently,

$$\lim_{x \to a} f(x) = L,$$

and say, the function f has **limit** L at $x = a$, to mean that the value of $f(x)$ approaches L as x approaches a.

Intuitively, we can make $f(x)$ as close as we like to L, simply by choosing x close enough to a. This is the essential idea captured in Cauchy's formal definition:

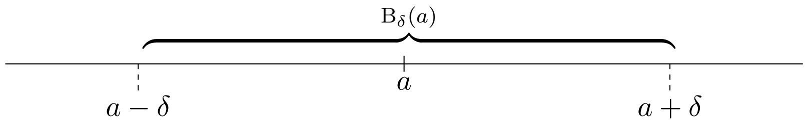
Definition 14.9.1. The function f has **limit** L at $x = a$ if

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0 \text{ such that } (0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon).$$

It's helpful to introduce *ball* notation here to simplify the statement.

If $a \in \mathbb{R}, \delta > 0$ then the **δ-neighbourhood** (abbreviated to **δ-nhd** of a, or the **δ-ball** centred at a is

$$\mathrm{B}_\delta(a) = (a - \delta, a + \delta) = \{x \in \mathbb{R} : |x - a| < \delta\}.$$



Also, the **punctured δ-neighbourhood** of a, or **punctured δ-ball** centred at a is

$$\mathrm{B}^*_\delta(a) = \mathrm{B}_\delta(a) \setminus \{a\} = \{x \in \mathbb{R} : 0 < |x - a| < \delta\} = (a - \delta, a) \cup (a, a + \delta).$$

Using the ball notation, we can write, f has limit L at $x = a$ if

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0 \text{ such that } f\big(\mathrm{B}^*_\delta(a)\big) \subseteq \mathrm{B}_\varepsilon(L)$$

which says, for all positive ε, as small as we like, there is a positive δ such that the punctured δ-neighbourhood of a maps under f entirely within an ε-neighbourhood of L.

By using the formal definition of *limit,* one can prove the following Limit Rules.

Algebra of Limits Laws. Suppose $f(x) \to L$ and $g(x) \to M$ as $x \to a$. Then:

1. $(f + g)(x) \to L + M$,
2. $(f - g)(x) \to L - M$,
3. $(f \cdot g)(x) \to L \cdot M$,
4. $\left(\frac{f}{g}\right)(x) \to \frac{L}{M}$, so long as $M \neq 0$.

Example. *1.* $\lim_{x\to 3} x^2 + 2x + 4 = 19$.

For polynomials there are no surprises. Limits can be evaluated by direct substitution.

2. Evaluate $\lim_{x\to 2} \dfrac{x^2-4}{x-2}$.

Solution.

$$\begin{aligned}\lim_{x\to 2}\frac{x^2-4}{x-2} &= \lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}\\ &= \lim_{x\to 2}(x+2)\\ &= 4.\end{aligned}$$

□

Note that everywhere except at $x = 2$, we have:

$$\frac{(x-2)(x+2)}{x-2} = x + 2,$$

i.e. the expression $x - 2$ can be cancelled everywhere except at $x = 2$, where it is zero. However, we can get as close as we like to $2 + 2 = 4$ by getting sufficiently close to $x = 2$. Using the formal definition, one can prove this. This legitimises the cancelling of expressions such as the $(x - 2)$ here, when evaluating limits.

3. $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$.

This is a standard limit. One can show that for small θ that $\sin\theta \approx \theta$, and that the error in making this approximation is bounded by $\frac{1}{3}|\theta|^3$.

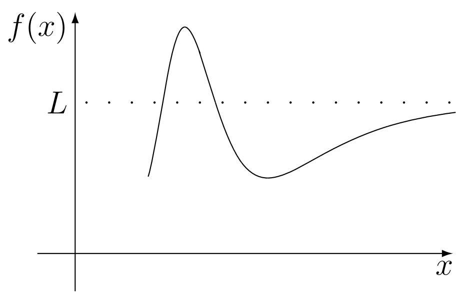
Note that as mentioned earlier, once one delves into Analysis, the trigonometric functions are defined to have arguments in radians.

14.10 Limits at Infinity

Given a function f defined for all large values of x, we write $f(x) \to L$ as $x \to \infty$, or equivalently,

$$\lim_{x\to\infty} f(x) = L,$$

to indicate that we can make the values of $f(x)$ as close to L as we like by making x sufficiently large.



As before there is a formal definition involving ε:

Definition 14.10.1. The function f has **limit** L **at infinity** if

$$\forall\, \varepsilon > 0\ \exists\, K \text{ such that } (x > K \implies |f(x) - L| < \varepsilon).$$

Now we said before that ∞ is not a real number. So it might surprise you that in certain circumstances we like to write:

$$\lim_{x\to a} f(x) = \infty.$$

Doesn't this mean that the limit doesn't exist? The answer is: Yes, but in a certain "regular" way, that we can use, and so it is convenient to have a short way to express it. For finite a, this notation is usually used for *one-sided* limits, i.e. instead of a one has either a^+ or a^- (where x approaches a from the *right* or *left*, respectively).

For now we will content ourselves with defining *infinite limits at infinity:*

Definition 14.10.2. The function f has **limit** ∞ **at infinity** if

$$\forall A > 0 \, \exists K \text{ such that } (x > K \implies f(x) > A).$$

In this case, we write: $f(x) \to \infty$ as $x \to \infty$ or, equivalently: $\lim_{x\to\infty} f(x) = \infty$.

There are similar definitions for limits as $x \to -\infty$ and with limit $-\infty$. Many of the *Algebra of Limits Laws* still work with each of L, M or a equal to either ∞ or $-\infty$ except where the "limit" would be $\infty - \infty$ or ∞/∞; in these cases, *Algebra of Limits* is inconclusive, though some other approach may provide a conclusive result. We give some examples of the possibilities.

Example 14.10.3. *1. If $f(x) \to L$, where L is finite, and $g(x) \to \infty$ as $x \to a$ then $(f + g)(x) \to \infty$ and $\left(\frac{f}{g}\right)(x) \to 0$ as $x \to a$.*

2. If $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$ then $(f - g)(x)$ may approach any finite or infinite limit as $x \to a$, e.g. consider the limits as $x \to \infty$ for $f(x) = x^2$ with each of the following possibilities for $g(x)$:

(i) $g(x) = x$ *(ii)* $g(x) = 2x^2$ *(iii)* $g(x) = x^2 - c$, $c \in \mathbb{R}$

3. If $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$ then $\left(\frac{f}{g}\right)(x)$ may approach any finite or infinite limit as $x \to a$, e.g. consider the limits as $x \to \infty$ for $f(x) = x^2$ with each of the following possibilities for $g(x)$:

(i) $g(x) = x$; *here* $\left(\frac{f}{g}\right)(x) \to \infty$ *as* $x \to \infty$;

(ii) $g(x) = x^3$; *here* $\left(\frac{f}{g}\right)(x) \to 0$ *as* $x \to \infty$;

(iii) $g(x) = 2x^2 - 1$; *here* $\left(\frac{f}{g}\right)(x) \to \frac{1}{2}$ *as* $x \to \infty$.

14.11 Continuous Functions

One would think the **continuous** functions ought to be those functions whose graphs one could draw without lifting one's pen from the page, and yes *continuous functions* have that property, but we define *continuous* in terms of *limits:*

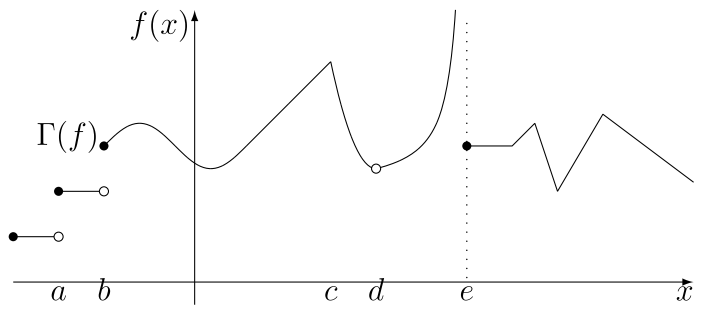
Definition 14.11.1. We say a function f is **continuous** *at a point* $x = a$ if

(i) the limit of f at $x = a$ exists, i.e. $\lim_{x\to a} f(x) = L$, say, and

(ii) $f(a) = L$.

We say a function f is **continuous** *in an open interval* I if f is *continuous* at every point in I. If $I = \mathbb{R}$ then we simply say f is *continuous.*

Consider the graph $\Gamma(f)$, below.



The function f has a **jump discontinuity** at each of $x = a$ and $x = b$. There is a hole in the graph at $x = d$, which can be removed by adding d to the domain of f and defining $f(d)$ to “fill in the hole”, i.e. by defining $f(d) = \lim_{x\to d} f(x)$; the point at d is called a **removable singularity** of f. Redefining f in this way at d, makes f *continuous* at d. At $x = e$ there is an **essential singularity** of f since $f(x) \to \infty$ as x approaches e from below (written $x \to e^-$), whereas $f(x)$ approaches a finite number as x approaches e from above, i.e. $\lim_{x\to e^+} f(x)$ is finite. This exemplifies one way a function can fail to be continuous, i.e. when *left* and *right* limits exist at a point but are different.

Example 14.11.2. *1. Every polynomial function is continuous.*

2. *The trigonometric functions* sin *and* cos *are continuous.*
3. *The floor and ceiling functions are discontinuous at each integer point; each discontinuity is a* jump discontinuity.

If f and g are continuous at $x = a$, then from the Algebra of Limits Laws we can deduce that $f + g$, $f - g$ and $f \cdot g$ are continuous at $x = a$, and $\frac{f}{g}$ is continuous at $x = a$, if $g(a) \neq 0$.

Also, the composite function $f \circ g$ is continuous at a, if g is continuous at a and f is continuous at $g(a)$.

14.12 Bolzano's Theorem

Theorem 14.12.1 (Bolzano). *If f is continuous on $[a, b]$, and $f(a)$ and $f(b)$ have different signs then $\exists c \in (a, b)$ such that $f(c) = 0$.*

Example 14.12.2. *1. Every odd degree polynomial $p(x)$ has a zero, since*

$$\lim_{x\to\infty} p(x) \text{ and } \lim_{x\to-\infty} p(x)$$

are both infinite but of opposite sign.

2. *For any $n \in \mathbb{N}$ such that $n > 1$, the real n^{th} root of 2 lies strictly between 0 and 2, since the polynomial $p(x) = x^n - 2$ is continuous on $[0, 2]$, and $p(0) = -2$ and $p(2) = 2^n - 2 > 0$.*

14.13 Intermediate Value Theorem

Theorem 14.13.1 (Intermediate Value Theorem). *If f is continuous on $[a, b]$, $f(a) \neq f(b)$ and M lies between $f(a)$ and $f(b)$, then $\exists c \in (a, b)$ such that $f(c) = M$.*

The *Intermediate Value Theorem* follows from and generalises *Bolzano's Theorem.*

14.14 Extreme Value Theorem

Theorem 14.14.1 (Extreme Value Theorem). *If f is continuous on $[a, b]$ then $\exists c_1, c_2 \in [a, b]$ such that $f(c_1) \leqslant f(x) \leqslant f(c_2)\, \forall x \in [a, b]$.*

From this last result it follows that a function that is continuous on a closed interval attains both its maximum and minimum on that interval. The statement is false for open intervals, e.g. $f(x) = \frac{1}{x}$ with domain $(0, 1)$ has neither a minimum nor maximum in $(0, 1)$.

14.15 Sequences and Series

Definition 14.15.1. If for each $n \in \mathbb{N}$, $a_n \in \mathbb{R}$ then the infinite list of terms

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

which may be abbreviated to (a_n) or more explicitly $(a_n)_{n=1}^{\infty}$, is called a **sequence**.

Thus, the terms of a *sequence* are just the values $a_n = f(n)$ of a *function* f whose domain is $\mathbb{N}$.

The sequence (a_n) **converges** *(is convergent)* if $\exists L \in \mathbb{R}$ such that

$$\lim_{n\to\infty} a_n \Big(= \lim_{n\to\infty} f(n)\Big) = L.$$

We also write, in this case, $a_n \to L$ as $n \to \infty$.

There is a formal definition for the **limit** of a *sequence* that is similar to the formal definition of *limit* for a function:

A *sequence* (a_n) has **limit** L if

$$\forall \varepsilon > 0 \, \exists N \in \mathbb{N} \text{ such that} (n > N \implies |a_n - L| < \varepsilon).$$

A sequence that is *not convergent,* is said to be **divergent**.

Example 14.15.2. *1. As* $n \to \infty$, $a_n = r^n \begin{cases} \to 0, & \textit{if } |r| < 1 \\ \to 1, & \textit{if } r = 1 \\ \textit{diverges}, & \textit{otherwise.} \end{cases}$

2. As $n \to \infty$, $a_n = \dfrac{1}{n^p} \begin{cases} \to 0, & \textit{if } p > 0, \\ \to 1, & \textit{if } p = 0, \\ \textit{diverges}, & \textit{if } p < 0. \end{cases}$

3. $a_n = \left(1 + \frac{x}{n}\right)^n \to e^x$ *as* $n \to \infty$.

A sequence (a_n) is said to be **increasing** if $a_{n+1} \geqslant a_n \; \forall n \in \mathbb{N}$, and to be **decreasing** if $a_{n+1} \leqslant a_n \; \forall n \in \mathbb{N}$; (a_n) is **monotone** if it is either *increasing* or *decreasing.*

Example 14.15.3. *1. If* $a_n = \dfrac{1}{n}$*, then* (a_n) *is* decreasing.

2. If $a_n = n^2 + n$*, then* (a_n) *is* increasing.

3. If $a_n = (-1)^n$*, then* (a_n) *is* alternating *(i.e. its sign alternates, for increasing* n*);* (a_n) *is* not *monotone.*

Definition 14.15.4. If for each $n \in \mathbb{N}$, $a_n \in \mathbb{R}$ then an infinite *sum* of terms

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which may be abbreviated to $\sum a_n$ or more explicitly $\sum_{n=1}^{\infty} a_n$, is called a **series**.

Of primary interest is when a *series* has a *finite* sum, i.e. *converges*. The *convergence* of a *series* is defined in terms of its **sequence of partial sums**, namely the sequence (s_n) where s_n is the n^{th} **partial sum** of $\sum a_n$, given by,

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

If s is finite and $s_n \to s$ as $n \to \infty$ then we say that $\sum a_n$ **converges** to s and write $\sum_{n=1}^{\infty} a_n = s$. If (s_n) diverges then $\sum a_n$ is said to **diverge**.

Theorem 14.15.5. *If $\sum a_n$ converges then $a_n \to 0$.*

Proof. Suppose the $\sum a_n$ converges then the partial sums s_n and s_{n-1} converge to the same limit L, say, as $n \to \infty$.
Hence $a_n = s_n - s_{n-1} \to L - L = 0$, as $n \to \infty$. □

The above theorem is usually used in its *contrapositive* form:

Theorem 14.15.6. *If $a_n \not\to 0$ then $\sum a_n$ does not converge.*

Example 14.15.7. *1.* $\sum \dfrac{n^2}{2n^2+n}$ *diverges since* $a_n = \dfrac{n^2}{2n^2+n} \to \frac{1}{1}2 \neq 0$.

Testing whether $a_n \to 0$ is such an easy check, that it should be the *first* thing one checks for. Of course, if $a_n \to 0$ *no* conclusion may be made, unless the series has some other property, e.g. it's *geometric* (then it converges) or it's an *alternating series* (then it converges).

2. If $a_n = ar^n$ *then* $\sum a_n = \sum_{n=1}^{\infty} ar^{n-1}$ *is a* ***geometric series****, for which the* n^{th} partial sum *is*

$$s_n = \frac{a(1-r^n)}{1-r}$$

which converges *to* $\frac{a}{1-r} \iff |r| < 1$.

3. $\sum \frac{1}{n}$ *is the* ***harmonic series****. It* diverges *since the* $(2^{m+1})st$ partial sum

$$\begin{aligned} & 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^m+1} + \cdots + \frac{1}{2^{m+1}}\right) \\ \geqslant\ & 1 + \frac{1}{2} + \frac{1}{2} \qquad + \frac{1}{2} \qquad + \cdots + \frac{1}{2} \\ =\ & 1 + \frac{m+1}{2} \to \infty \text{ as } m \to \infty. \end{aligned}$$

4. $\sum \frac{1}{n^p}$ *(a* ***p-series****) converges* $\iff p > 1$.

The *harmonic series* is the case $p = 1$. Essentially this result says that if the terms of a *p-series* grow more slowly (i.e. $p > 1$) than the *harmonic series* then it converges. Otherwise, the terms grow at least as fast as the *harmonic series* and the series diverges.

14.16 Comparison Tests

Theorem 14.16.1 (Comparison Test). *If for some* $N \in \mathbb{N}$, $0 < a_n < b_n \; \forall n > N$, *then*

(i) $\sum b_n$ *converges* $\implies$ $\sum a_n$ *converges; and*

(ii) $\sum a_n$ *diverges* $\implies$ $\sum b_n$ *diverges.*

Theorem 14.16.2 (Limit Comparison Test). *If* $\lim_{n\to\infty} \frac{a_n}{b_n}$ *exists, then* $\sum a_n$ *and* $\sum b_n$ *either* both converge *or* both diverge.

Example 14.16.3.

1. $\sum \frac{1}{2n!}$ *converges since* $\frac{2^n}{2n!} \to 0$ *and* $\sum \frac{1}{2^n}$ *is a convergent* geometric series.
2. $\sum \frac{1}{\sqrt{n(n+1)}}$ *diverges since* $\frac{n}{\sqrt{n(n+1)}} \to 1$ *and* $\sum \frac{1}{n}$ *(the* harmonic series*) diverges.*

14.17 The Ratio Test

Theorem 14.17.1 (Ratio Test). *Suppose* $\left|\frac{a_{n+1}}{a_n}\right| \to \rho$. *Then*

$$\sum a_n \begin{cases} \text{converges,} & \text{if } \rho < 1 \\ \text{diverges,} & \text{if } \rho > 1. \end{cases}$$

If $\rho = 1$ *the test is inconclusive.*

For a *geometric series* $\sum ar^{n-1}$, $\rho = r$. Essentially, the *Ratio Test* determines whether $\sum a_n$ behaves like a *geometric series.* The comparison is too weak if the ratio is 1, and so then no conclusion is possible, unless $\sum a_n$ *is* a *geometric series;* in which case, one can conclude immediately that $\sum a_n$ diverges. The lesson here is: don't use the *Ratio Test* if the series is a *geometric series,* since that knowledge already *conclusively* tells you whether or not the series converges.

Example 14.17.2. *1.* $\sum \frac{n!}{n^n}$ *converges since*

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \left(\frac{n}{n+1}\right)^n \to \frac{1}{e} < 1 \text{ as } n \to \infty.$$

2. $\sum \frac{n!}{2^n}$ *diverges since*

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{2^{n+1}} \cdot \frac{2^n}{n!} = \frac{n+1}{2} \to \infty \text{ as } n \to \infty.$$

We noted earlier that the *first* thing we should check is whether the n^{th} term a_n goes to 0. Here, $a_n \to \infty$. So, already we have that $\sum a_n$ diverges! Using the *Ratio Test* here is wasted effort. *Always* check the easy tests first.

14.18 Functional Equations

A **functional equation** is an equation involving an unknown function f and variables such as x and y.

In order to find a *solution* of a *functional equation* one must usually find *explicit* expressions $f(x)$.

Example 14.18.1. 1. $f(xy) = y^k f(x)$

Solution. *Substituting* $x = 1$*, gives*

$$f(y) = y^k f(1)\ \forall y \in \mathbb{R}.$$

Letting $c = f(1)$*, and changing the dummy variable to* x*, we have that solutions of 1. must be of form*

$$f(x) = cx^k \text{ for some constant } c\ \forall x \in \mathbb{R}.$$

Now we check that all such f *are solutions. Assume* $f(x) = cx^k$ *for some constant* c*. Then*

$$\begin{aligned} f(xy) &= c(xy)^k \\ &= y^k cx^k \\ &= y^k f(x), \end{aligned}$$

as required.

2. $f(x + y) = f(y)$

Solution. *Substituting* $y = 0$*, we have*

$$f(x) = f(0).$$

Letting $c = f(0)$*, we have that solutions of 2. must be of form*

$$f(x) = c \text{ for some constant } c\ \forall x \in \mathbb{R}.$$

Now assume $f(x) = c$ *for some constant* c*. Then*

$$\begin{aligned} f(x + y) &= c \\ &= f(y). \end{aligned}$$

Thus the solutions of 2. are the constant functions: $f(x) = c$*,* c *constant, for all* $x \in \mathbb{R}$*.*

3. $f(x + y) = f(x) + f(y)$ *(Cauchy's equation)*

Solution. *Letting* $y = x$ *in 3. we obtain* $f(2x) = f(x) + f(x) = 2f(x)$*. Then letting* $y = 2x$ *in 3. we get*

$$f(3x) = f(x) + f(2x) = 3f(x). \tag{14.18.1}$$

Thus, it is apparent that by induction we can show that

$$f(nx) = nf(x)\ \forall n \in \mathbb{N}. \tag{14.18.2}$$

Indeed, defining the proposition

$$P(n) : f(nx) = nf(x),$$

we have $P(1) : f(x) = f(x)$*, trivially, and* $P(k) \implies P(k+1)$ *is immediate, since* (14.18.1) *can be generalised, by replacing* 2 *by* k *and* 3 *by* $k + 1$*. Thus* (14.18.2) *follows.*

Substituting $x = 1$, *in* (14.18.2), *we have*

$$f(n) = nf(1)\ \forall n \in \mathbb{N}.$$

Set $c = f(1)$. *Then we have*

$$f(n) = nc\ \forall n \in \mathbb{N}. \tag{14.18.3}$$

Now substituting $x = \frac{m}{n}$ *(so that* $nx = m$*), where* $m, n \in \mathbb{N}$, *in* (14.18.2), *we have*

$$\begin{aligned} nf(x) &= f(nx) \\ &= f(m) = mc, \qquad \text{by (14.18.3)} \\ \therefore f(x) &= \frac{m}{n}c = xc\ \forall x \in \mathbb{Q}^+ \end{aligned} \tag{14.18.4}$$

Substituting $x = y = 0$ *in 3. we have*

$$\begin{aligned} f(0) &= f(0) + f(0) \\ \therefore 0 &= f(0). \end{aligned}$$

Thus, now substituting $y = -x$ *in 3. we have*

$$\begin{aligned} 0 &= f(0) \\ &= f(x + -x) \\ &= f(x) + f(-x) \\ \therefore f(-x) &= -f(x) \\ &= -(xc) = (-x)c \\ \therefore f(x) &= \frac{m}{n}c = xc, c \text{ constant } \forall x \in \mathbb{Q} \end{aligned}$$

If f *is also continuous for all* $x \in \mathbb{R}$, *then, by considering rational sequences that converge to each irrational* $x \in \mathbb{R}$ *(and taking limits), we have*

$$f(x) = xc, c \text{ constant } \forall x \in \mathbb{R}.$$

4. $f(x + y) = f(x)f(y)$

Solution. *Letting* $x = y = 0$ *in 4. we have*

$$\begin{aligned} f(0) &= f(0) \cdot f(0) \\ f(0)\big(1 - f(0)\big) &= 0 \\ \therefore f(0) &= 0 \text{ or } 1. \end{aligned}$$

Let $y = 0$ *in 4. and suppose* $f(0) = 0$. *Then* $f(x) = f(x)f(0) = 0$, *i.e.* $f(x) = 0$ *for all* $x \in \mathbb{R}$. *Checking, we find* $f(x) = 0$ *satisfies 4.*
Now let $f(0) = 1$.
Letting $y = x$ *in 4. we obtain* $f(2x) = f(x)f(x) = f(x)^2$. *Then letting* $y = 2x$ *in 4. we get*

$$f(3x) = f(x)f(2x) = f(x)^3.$$

Thus, by a similar induction to 3. we can deduce

$$f(nx) = f(x)^n\ \forall n \in \mathbb{N}. \tag{14.18.5}$$

Substituting $x = 1$*, in* (14.18.5)*, we have*

$$f(n) = f(1)^n \ \forall n \in \mathbb{N}.$$

Set $a = f(1)$*. Then we have*

$$f(n) = a^n \ \forall n \in \mathbb{N}. \tag{14.18.6}$$

Now substituting $x = \frac{m}{n}$ *(so that* $nx = m$*), where* $m, n \in \mathbb{N}$*, in* (14.18.2)*, we have*

$$\begin{aligned} f(x)^n &= f(nx) \\ &= f(m) = a^m, \qquad \text{by } (14.18.6) \end{aligned}$$

At this point, we would like to take the n^{th} *root, but this is only well-defined if* a^m *(and therefore* a*) is positive.*
So assume from now on that $a > 0$*. Then*

$$f(x) = a^{\frac{m}{n}} = a^x \ \forall x \in \mathbb{Q}^+.$$

Now substitute $y = -x$ *in 4. we have*

$$\begin{aligned} 1 &= f(0) \\ &= f(x + -x) \\ &= f(x) + f(-x) \\ \therefore f(-x) &= \frac{1}{f(x)} \\ &= \frac{1}{a^x} = a^{-x} \\ \therefore f(x) &= a^x, a \text{ a positive constant } \forall x \in \mathbb{Q} \end{aligned}$$

If f *is also* continuous *for all* $x \in \mathbb{R}$*, then, by considering rational sequences that converge to each irrational* $x \in \mathbb{R}$ *(and taking limits), we have*

$$f(x) = a^x, a \text{ a positive constant } \forall x \in \mathbb{R}.$$

Thus we see that if f *is required to be continuous and have domain* $\mathbb{R}$*, then either*

$$f(x) = 0 \ \forall x \in \mathbb{R} \quad \text{or} \quad f(x) = a^x, \ a > 0, \forall x \in \mathbb{R}.$$

Earlier we wrote $\exp_a$ *for the function* $f : x \mapsto a^x$*,* $a > 0$*. All such functions can be written in terms of* $\exp = \exp_e$*,*

$$\exp_a(x) = a^x = (e^{\ln a})^x = e^{cx} = \exp(cx), \text{ where } c = \ln a.$$

5. $f(xy) = f(x) + f(y)$

Solution. *If* f *is defined at* 0*, then letting* $x = y = 0$ *in 5. we have*

$$\begin{aligned} f(0) &= f(0) + f(0) \\ 0 &= f(0). \end{aligned}$$

Now let $y = 0$ *in 5. Then*

$$0 = f(0) = f(x) + f(0).$$

Thus if 0 *is in the domain of* f*, then*

$$f(x) = 0 \ \forall x \in \mathbb{R}.$$

Checking, we see that this does indeed satisfy 5.

To get other solutions, we need $0 \notin \mathrm{dom}(f)$. If $\mathrm{dom}(f) = (0, \infty)$ and f is continuous on this domain, then

$$f(x) = c \log x, \text{ for constant } c \in \mathbb{R}.$$

If $\mathrm{dom}(f) = (-\infty, 0)$ with f is continuous on this domain, then

$$f(x) = c \log |x|, \text{ for constant } c \in \mathbb{R}$$

is a family of solutions.

6. $f(xy) = f(x)f(y)$

Solution. *Let $y = 0$ in 6. Then*

$$\begin{aligned} f(0) &= f(x)f(0) \\ f(0)\big(1 - f(x)\big) &= 0 \\ \therefore f(0) &= 0 \text{ or } f(x) = 1 \ \forall x \in \mathbb{R}. \end{aligned}$$

Let $y = 1$ in 6. Then

$$\begin{aligned} f(x) &= f(x)f(1) \\ f(x)\big(1 - f(1)\big) &= 0 \\ \therefore f(1) &= 1 \text{ or } f(x) = 0 \ \forall x \in \mathbb{R}. \end{aligned}$$

Suppose that for some x, say $x = a$, that $f(a) \neq 0$ and $f(a) \neq 1$. Then for $y = a^n$, $n \in \mathbb{N}$, we can show by induction that

$$f(a^n) = f(a)^n \ \forall n \in \mathbb{N}.$$

Now suppose $x = e^n > 0$ and suppose $f(e) = b > 0$ and $b \neq 1$. Then

$$f(x) = f(e^n) = f(e)^n = b^n = (e^{\ln b})^n = (e^n)^{\ln b} = x^{\ln b}.$$

Thus, if we let $c = \ln b$ then we have solutions of form

$$f(x) = x^c, x = e^n > 0, n \in \mathbb{N}.$$

Arguments similar to those in 3.–5. with f assumed continuous lead to

$$f(x) = x^c, x > 0.$$

Checking, we find that each of the solutions we have found,

$$f(x) = 0 \ \forall x \in \mathbb{R}, f(x) = 1 \ \forall x \in \mathbb{R}, f(x) = x^c, c \in \mathbb{R} \ \forall x \in (0, \infty),$$

satisfy 6. The last of these can be extended to include 0 in $\mathrm{dom}(f)$ if $c > 0$.

7. $f\big(\frac{x+y}{2}\big) = \frac{1}{2}\big(f(x) + f(y)\big)$ *(Jensen's equation)*

Solution. *Let $f(0) = b$ and $f(1) = a + b$ for some $a, b \in \mathbb{R}$. Then put $y = 0$ and consider $x = n \in \mathbb{N}$, and then arguments similar to those in 3. with continuity of f assumed, lead to:*

$$f(x) = ax + b, a, b \in \mathbb{R}, \ \forall x \in \mathbb{R}.$$

Exercise Set 14.

1. Determine all monotonic involutions.

   Note that **monotonic** is synonymous with *monotone*. An **involution** is a function f that satisfies
   $$f\big(f(x)\big) = x,$$
   for all x, i.e. a function that is its own inverse.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that the n^{th} iterate f^n has a unique fixed point x_0. Prove that f has a unique fixed point x_0.

3. (1994 AMOC Senior Contest Q5) Let $f : \mathbb{R} \to \mathbb{R}$ is a function such that for all $x \in \mathbb{R}$,

   (i) $f(x) \neq 0$, and
   (ii) $f(x+2) = f(x-1)f(x+5)$.

   Prove that f is **periodic**, i.e. there exists $P > 0$ such that $f(x+P) = f(x)$ for all x.

4. If $f : [0,1] \to [0,1]$ is continuous, prove that $f(x) = x$ for some x.

5. Prove there are 2 points on opposite sides of the Earth with exactly the same temperature.

6. Prove that $e^x = x^2$ has a real solution.

7. Find all $x \in \mathbb{R}$ such that $x + \sqrt{x-5} > 3$.

8. Solve $|x-a| + |x-b| = k$ for all real a, b, k.

9. Show that there is a solution $f(x) = 0$ between 0 and 1, given $f(x) = 4x^2 + x - 2$.

10. Find all $x \in \mathbb{R}$ such that $\sqrt[3]{x+2} + \sqrt[3]{x+3} + \sqrt[3]{x+4} = 0$.

11. Determine all real solutions of the following system of simultaneous equations:
    $$a^2 + b^2 = 6c,\ b^2 + c^2 = 6a,\ c^2 + a^2 = 6b.$$

12. Given a continuous function $f : [0,1] \to \mathbb{R}$ with $f(0) = f(1)$, for what d does there exist an x such that $f(x) = f(x+d)$.

13. Suppose $x + y + z = 5$ and $xy + yz + zx = 3$. Show that x, y, z lie between -1 and $\frac{13}{3}$.

14. Let $n \in \mathbb{N}$. Determine how many $x \in \mathbb{R}$ with $1 \leqslant x < n$ satisfy $x^3 = \lfloor x \rfloor^3 + \{x\}^3$, where $\{x\} = x - \lfloor x \rfloor$.

15. Find all $x \in \mathbb{R}$ such that $(x+2010)(x+2011)(x+2012)(x+2013) + 1 = 0$.

16. Let $f(x) = 5^x$. Find all $x \in \mathbb{R}$ such that $f\big(x + f(2008)\big) = 2008 - x$.

17. Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that

    (a) $f(x+y) = f(x) + f(y)$
    (b) $f(x+y) = f(x)f(y)$
    (c) $f(xy) = f(x) + f(y)$

18. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f\big(x - f(y)\big) = 1 - x - y$ for all $x, y \in \mathbb{R}$.