CHAPTER 16

Graph Theory

Graph Theory was invented in 1736, when Leonhard Euler solved the *Königsberg Bridge Problem* (see Exercise 19). In older texts, the diagram that Euler used to solve the problem was referred to as a *graph*; the more modern term (that we use) is *multigraph.* And what we refer to as a *graph*, in older texts was referred to as a **simple graph**.

Since the terminology is extensive, we start with a core set of definitions.

16.1 Introductory definitions

multigraph, G, consists of a pair V, E, where E is a set of unordered pairs of elements of V.

The elements of V are called **vertices** (singular: **vertex**).
The elements of E are called **edge**s.
An unordered pair of vertices of V, that is *not* an element of E is called a **non-edge**.

A *multigraph* is realised as a diagram by representing the *vertices* as points, with a line segment joining pairs of vertices u, v if and only if $\{u, v\}$ is an *edge.*

A *multigraph* may have more than one edge joining a pair of its vertices, but all its edges join distinct vertices, i.e. a *multigraph* has no *loops* (see below).

If $e = \{u, v\}$ (which may be abbreviated to uv or vu) is an *edge* of a graph G, then u and v are said to be **adjacent**, and u, v are called **endpoints** (or **ends**, or **terminals**) of e.

incident. If $e = uv$ is an *edge* then vertices u, v are said to be **incident** with e, and also e is said to be **incident** with each of u and v.

loop. An edge whose endpoints are *not* distinct.

pseudograph, G, consists of a *vertex* set V and *edge* set E, where E may contain *loops.*

graph. A *multigraph* (i.e. it has no *loops*) that has at most one *edge* joining any pair of vertices.

degree, of a vertex v, written $\partial(v)$, is the number of edges it is incident with.

order, of a graph G, is the cardinality of its *vertex* set.

size, of a graph G, is the cardinality of its *edge* set.

(p, q) **graph.** A graph of *order* p and *size* q, i.e. a graph with p *vertices* and q *edges.*

16.1.1 Convention when drawing graphs

Vertices are indicated by filled in dots. So, if two edges cross and there is *no* filled in dot at the crossing point of those edges, then the edges do not intersect and there is not a vertex at the crossing point of those edges.

subgraph. A *graph* H is a **subgraph** of a *graph* G, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

If e is an *edge* of G, then $G - e$ is the *subgraph* of G whose vertex set is $V(G)$ and whose edge set is $E(G) \setminus \{e\}$.

If v is a *vertex* of G, then $G - v$ is the *subgraph* of G whose vertex set is $V(G) \setminus \{v\}$ and whose edge set consists of all the edges of G except those incident with v.

complement. The **complement** $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ and edge set, the set of *non-edges* of G.

The union of G and its complement $\overline{G}$ is the complete graph with $|V|$ vertices.

digraph, directed graph, D, consists of a pair V, E, where E is a set of *ordered* pairs of elements of V. As with a *graph*, V is the *vertex set* of D. The elements of E are called **arcs** (or **directed edges**). An *arc* $e = (u, v)$ of D is represented by a line segment joining the pair of vertices u, v with an *arrow* in the direction from u to v.

even vertex. A *vertex* of *even degree.*

odd vertex. A *vertex* of *odd degree.*

leaf. A vertex of degree 1.

isomorphic. Two graphs G_1 and G_2 are **isomorphic** if there is a one-to-one map from $V(G_1)$ to $V(G_2)$ that preserves adjacency.

walk. A u-v **walk** is an alternating sequence $u = u_1, e_1, u_2, e_2, u_3, \ldots, u_n = v$ of vertices and edges of a graph such that vertices u_i and u_{i+1} are incident with edge e_i for $1 \leqslant i < n$.

closed walk. A u-v **walk** for which $u = v$.

trail. A u-v **trail** is a u-v *walk* that does not pass through the same *edge* twice.

path. A u-v **path** is a u-v *walk* that does not pass through the same *vertex* twice, except that possibly $u = v$.

circuit. A u-v *trail* with at least three vertices, for which $u = v$.

cycle. A *circuit* that does not repeat a *vertex,* except for the first and last vertices.

Euler trail, Euler tour. A *trail* of a multigraph G that passes through every edge of G *exactly once.*

Euler circuit. An *Euler trail* of a multigraph G that is a *circuit.*

Euler multigraph. A multigraph G for which an *Euler trail* exists.

Hamiltonian path. A u-v path that passes through each vertex of a multigraph exactly once, except that possibly $u = v$.

Hamiltonian circuit. A Hamiltonian path that is a circuit.

Hamiltonian multigraph. A multigraph G for which an *Hamiltonian path* exists.

regular. A graph is r**-regular** if every vertex is of *degree* r.

complete graph, K_n, is an $(n-1)$*-regular* graph, i.e. every pair of its vertices are adjacent.

connected. Two *vertices* u, v in a graph G are **connected** if $u = v$ or a *u-v path* exists in G.

A *graph* G is *connected* if every pair of vertices of G is *connected*.

component. A *subgraph* H of a *graph* G that is maximal with respect to the property of being *connected*, i.e. H is not contained in any connected subgraph of G having more vertices or edges.

cut-vertex, of a *connected graph* G is a *vertex* v of G, such that $G - v$ is *not connected*.

bridge, of a *connected graph* G is an *edge* e of G, such that $G - e$ is *not connected*.

tree. A *connected graph* that has no *cycles*.

forest. A *graph* that has no *cycles*, i.e. a *graph* whose *components* are *trees*.

spanning tree, of a *connected graph* G is a *subgraph* H of G that is a tree and such that $V(H) = V(G)$.

planar. A *graph* G is **planar** if one can draw G in the plane in such a way that no edges cross.

bipartite graph. A graph G for which it is possible to find a *partition* of its vertex set $V(G)$ into two sets X, Y such that every edge in $E(G)$ joins a vertex in X to a vertex in Y.

colouring, of a graph G is an an assignment of *colours* (i.e. labels) to the vertices of G such that each pair of adjacent vertices are assigned *different* colours.

An n**-colouring** is a *colouring* of a graph G using n *colours*.

chromatic number, χ**.** The **chromatic number** $\chi(G)$ of a graph G is the *minimum* value n for which an n*-colouring* of G exists.

length, of a *path* P of a graph (resp. digraph) is the number of *edges* (resp. *arcs*) in P.

distance, $d(u, v)$ is the *length* of the *shortest u-v path*.

diameter, $\operatorname{diam}(G) = \max_{u,v \in V(G)} d(u, v)$.

clique. A *subgraph* of a graph that is complete.

tournament. A *digraph* for which each pair of vertices u, v exactly one of (u, v) or (v, u) is an arc.

Exercise Set 16.

1. Prove:

   Lemma (Handshaking Lemma). *For a (p, q)-graph whose vertices have degrees $d_1, d_2, \ldots, d_p$ (called a* ***degree sequence****),*

   $$\sum_{i=1}^{p} d_i = 2q.$$

   i.e. the sum of the degrees is twice the number of edges.

2. Using the previous result, why must the number of odd-degree vertices be even?

3. Show there is no graph with vertices of degrees 2, 3, 3, 4, 4, and 5.

4. Show there is no graph with vertices of degrees 2, 3, 4, 4, and 5.

5. Show there is no graph with vertices of degrees 1, 3, 3, and 3.

6. Give an example of a graph that

   (i) has no vertex of odd degree.
   (ii) has no vertex of even degree.
   (iii) has 4 components, 6 vertices and 3 edges.
   (iv) is 3-regular but is not complete.

7. (a) Let $m, n \in \mathbb{Z}$ such that $1 \leqslant m \leqslant n$. Give an example of a graph with n vertices and m components.
   (b) Is it possible for a graph to have more components than vertices? Explain.

8. Show that a tree of order n has size $n - 1$.

9. Suppose you and your partner attend a party with 3 other couples. Several handshakes take place. No one shakes hand with themselves or their partner, and no one shakes hand with the same person more than once. After the handshaking is completed, you ask each person including your partner, how many hands they had shaken, and each person gave a different answer.

   (a) How many hands did you shake?
   (b) How many hands did your partner shake?

10. Show that in any group of at least two people, there are always two people with the same number of friends.

11. Let G be a graph of order n. Show that if every vertex of G has degree at least $\frac{1}{2}(n-1)$ then G is connected.

12. Out of 6 boys, exactly 2 were known to have stolen apples. But whom? Albert dobbed in Billy and Charlie. Billy said that Edward and Donald stole the apples. Charlie dobbed in Albert and Billy. Donald said that it was Edward and Fred. Edward said that Fred and Billy were the thieves. Fred couldn't be found. Four of the boys questioned had nominated one thief correctly, but lied about the other. The fifth had lied outright.

    Who stole the apples?

13. Prove that among 6 people there are either 3 mutual friends or 3 mutual strangers.

14. Is it possible that at a party with 777 attendees, each attendee knows exactly 77 other attendees?

15. Prove the distance relationship:

$$d(u, v) \leqslant d(u, w) + d(v, w), \text{ for all vertices } u, v, w.$$

16. Prove that any graph with n vertices and n edges has a cycle.

17. Several lines divide the plane into regions. Prove that each region can be coloured black or white so that any pair of adjacent regions are coloured differently.

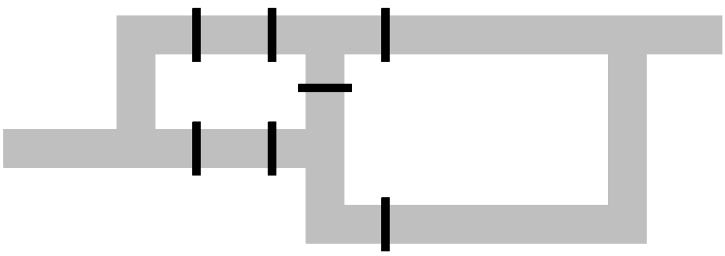
18. Prove that a connected graph is bipartite if and only if the graph has no odd cycle.

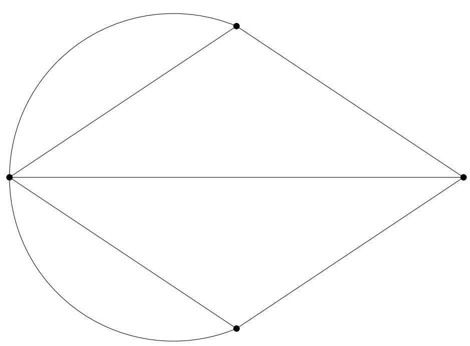
19. Prove:

> **Theorem (Euler circuit).** *A multigraph has an Euler circuit if every vertex has even degree.*
>
> In fact, the converse of the *Euler circuit theorem* is also true.

The above result was motivated by the **Königsberg bridge problem**:

> In the town of Königsberg in the 18th century there were seven bridges across the river Pregel. They connected two islands in the river with each other and with opposite banks. The townsfolk had long amused themselves with the problem:
>
> > Is it possible to cross the seven bridges in a continuous walk without recrossing any of them?





Leonhard Euler proved the above theorem in 1736, and hence solved the *Königsberg bridge problem.* Find Euler's solution.

The problem is equivalent to determining whether there is an Euler path for the following graph (each bridge is represented by an edge of the graph and the islands and banks of the river Pregel are represented by vertices of the graph).