CHAPTER 3

Mathematical Induction

3.1 Revisiting induction

We already visited *induction* as a proof technique in the last chapter. We describe it again; this time, in terms of a ladder. How might we prove we can climb a ladder?

One way to show we can climb a ladder is to show we can do two things:

(i) We can get on the *first* rung, and

(ii) we can get from any rung to the next rung.

Knowing we can do these two things, we can deduce that we can get to any rung of the ladder. This is essentially the *inductive* way to prove we can climb a ladder.

How might we express the above mathematically? Firstly, let $P(n)$ be the statement:

"we can get onto the n^{th} rung."

Now being able to climb the ladder is the same as saying

"we can get onto the n^{th} rung, for any n",

or, in terms of $P(n)$, that:

$P(n)$ is true, for any n.

Now, (i) above is: $P(1)$ is true.

And, another way of saying (ii) is that:

"***if*** we can get to the k^{th} rung, ***then*** we can get to the $(k+1)^{\text{st}}$ rung",

which in terms of $P(n)$ is:

$P(k) \implies P(k+1)$.

Lastly, mathematicians' ladders tend to have an *infinite* number of rungs. So *"for any n"* becomes: *"for all $n \in \mathbb{N}$"*.

Expressing the above formally, we have:

Definition 3.1.1. The **Principle of Mathematical Induction**, states:

To prove the statement $P(n)$ is true for all $n \in \mathbb{N}$, *it is sufficient* to prove:

(i) $P(1)$ is true, and
(ii) $P(k) \implies P(k+1)$, for general k.

Note 3.1.2. In fact, what we have described above is **simple induction**. Revisiting the ladder concept, we can imagine beasts with various afflictions that need to be catered for, and customise other forms of induction to show those beasts can still climb the ladder.

One way induction arises naturally is in proving a guessed relationship is true, e.g. from

$$\begin{aligned} 1 &= 1 \\ 1+2 &= 3 \\ 1+2+3 &= 6 \\ &\vdots \end{aligned}$$

we might guess that: $1+2+\cdots+n = \frac{1}{2}n(n+1)$, and then proceed to prove it by induction; this is done in the example below. We'll do the proof twice; the 1^{st} proof is accompanied by much explanation. The 2^{nd} proof omits superfluous explanation and follows protocols in Note 3.1.3.

Note 3.1.3. When we introduced *Mathematical Induction*, last chapter, we explained that, for readability, it's nice to begin an induction proof by defining a $P(n)$. Usually, you will be just asked to prove some statement in terms of an integer parameter n; introducing $P(n)$, effectively gives that statement a short name $P(n)$.

Then, it's tidy to have headings, *base case* (for step (i)), *inductive step* (for step (ii)), and a *conclusion* that states: "by **PMI**, from (i) and (ii) we deduce $P(n)$ is true for all $n \in \mathbb{N}$", where *PMI* abbreviates *Principle of Mathematical Induction*).

Also, it's a good practice to label the *assume* statement in the *inductive step*, **(I.A.)** (for **Inductive Assumption**), and to have *"by (I.A.)"* to the right of the step *(I.A.)* is applied. Finally, *"holds"* is a little nicer than *"is true"*.

Example 3.1.4. *Prove* $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ *for all* $n \in \mathbb{N}$.

Proof. First we let $\quad P(n) : 1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$.

Notice, a ':' was used here to indicate that $P(n)$ is short-hand for *everything* that follows the ':'. Use of a symbol like '=' instead of ':' would have been too confusing!

(i) Show $P(1)$ is true:

Proof. $P(1)$ is of the form LHS = RHS. To show it is true we start with one side and *reduce* it to the other side. Now the *LHS of P(1)* is just 1 and the *RHS of P(1)* is $\frac{1}{2} \cdot 1(1+1)$, i.e.

$$\begin{aligned} \text{LHS of } P(1) &= 1 \\ &= \tfrac{1}{2} \cdot 1(1+1) \\ &= \text{RHS of } P(1) \end{aligned}$$

So $P(1)$ is true. □

(ii) Show, for a general natural number k: $\quad$ if $P(k)$ is true then $P(k+1)$ is also true;

Proof. To prove a statement of form:

If *hypothesis* ***then*** *conclusion*

by Direct Proof (the most usual way, for an induction), we ***assume*** the *hypothesis* and deduce from it, the *conclusion.* Hence, we assume $P(k)$ is true, i.e. we assume

$$\text{LHS of } P(k) = \text{RHS of } P(k).$$

Now we wish to deduce that $P(k+1)$ is true, where $P(k+1)$ is of the form LHS = RHS. So to show $P(k+1)$ is true we start with one side and *reduce* it to the other side. (Somewhere along the way we expect to use our *assumption* that $P(k)$ is true – incidentally, this assumption is called the ***inductive assumption***). Thus, starting with one side ...

$$\begin{aligned} \text{LHS of } P(k+1) &= 1 + 2 + \cdots + k + k + 1 \\ &= \big(\text{LHS of } P(k)\big) + k + 1 \\ &= (\text{RHS of } P(k)) + k + 1, \quad \text{(using the inductive assumption)} \\ &= \tfrac{1}{2}k(k+1) + k + 1 \\ &= \tfrac{1}{2}(k+1)\big(k+2\big) \\ &= \tfrac{1}{2}(k+1)\big((k+1)+1\big) \\ &= \text{RHS of } P(k+1) \end{aligned}$$

So, *if* $P(k)$ *is true* then $P(k+1)$ is true. □

Thus we may now deduce from (i) and (ii), that, by *PMI*, $P(n)$ is true for all $n \in \mathbb{N}$. □

Now let's do the whole proof again, following the protocols of Note 3.1.3.

Proof. Let $P(n) : 1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$.

(base case) We prove $P(1)$.

$$\begin{aligned} \text{LHS of } P(1) &= 1 \\ &= \tfrac{1}{2} \cdot 1(1+1) \\ &= \text{RHS of } P(1) \\ \therefore\ P(1) \text{ holds.} & \end{aligned}$$

(inductive step) We prove: $P(k) \implies P(k+1)$ for general k.
Assume $P(k)$ holds, i.e. $1 + 2 + \cdots + k = \frac{1}{2}k(k+1)$ (I.A.)
Consider $P(k+1)$:

$$\begin{aligned} \text{LHS of } P(k+1) &= 1 + 2 + \cdots + k + k + 1 \\ &= \tfrac{1}{2}k(k+1) + k + 1, \quad \text{by (I.A.)} \\ &= \tfrac{1}{2}(k+1)\big(k+2\big) \\ &= \tfrac{1}{2}(k+1)\big((k+1)+1\big) \\ &= \text{RHS of } P(k+1) \\ \therefore\ P(k+1) \text{ holds, if } P(k) \text{ holds.} & \end{aligned}$$

(conclusion) So, we have proved:

$$\begin{aligned} &P(1) \text{ holds, and} \\ &P(k) \implies P(k+1), \text{for general } k, \end{aligned}$$

and hence, by PMI, $P(n)$ holds for all $n \in \mathbb{N}$, i.e.
$1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ for all $n \in \mathbb{N}$. ☐

Example 3.1.5. *Prove that:*

If $x + \dfrac{1}{x}$ *is an integer then* $x^n + \dfrac{1}{x^n}$ *is an integer for all positive integers* n.

You will notice differences between the structures of our proof below and that of our elementary example above, but you will notice also great similarities. One of the differences is that the ***inductive step*** *of the proof requires* two previous *"rungs", which means that the* ***base case*** *of the proof must be replaced with a proof that one* "can get onto the first two rungs" *– think of the* ladder. *This is one form of* ***secondary induction***.

Proof. Assume $x + \dfrac{1}{x} = N \in \mathbb{Z} \qquad (*)$

We show that $x^n + \dfrac{1}{x^n} \in \mathbb{Z}$ for all $n \in \mathbb{N}$, by induction.

Let $f(n) = x^n + \dfrac{1}{x^n}$.
Let $P(n) : f(n) \in \mathbb{Z}$.

(base case) We prove $P(1)$ and $P(2)$.

$$\begin{aligned} f(1) &= x^1 + \frac{1}{x^1} \\ &= x + \frac{1}{x} \\ &\in \mathbb{Z}, && \text{by } (*) \end{aligned}$$

$\therefore$ $P(1)$ holds.

$$\begin{aligned} f(2) &= x^2 + \frac{1}{x^2} \\ &= \left(x + \frac{1}{x}\right)^2 - 2x \cdot \frac{1}{x} \\ &= N^2 - 2 \\ &\in \mathbb{Z}, && \text{since by } (*),\ N \in \mathbb{Z} \end{aligned}$$

$\therefore$ $P(2)$ holds.

(inductive step) We prove: $P(k-1)$ and $P(k) \implies P(k+1)$ for $k \geqslant 2$.
Assume $P(k-1)$ and $P(k)$ hold, i.e. $f(k-1), f(k) \in \mathbb{Z} \qquad$ (I.A.)
Consider $P(k+1)$:

$$\begin{aligned} f(k+1) &= x^{k+1} + \frac{1}{x^{k+1}} \\ &= \left(x^k + \frac{1}{x^k}\right)\left(x + \frac{1}{x}\right) - x^k \cdot \frac{1}{x} - \frac{1}{x^k} \cdot x \\ &= \left(x^k + \frac{1}{x^k}\right)\left(x + \frac{1}{x}\right) - \left(x^{k-1} + \frac{1}{x^{k-1}}\right) \\ &= f(k) \cdot N + f(k-1), \quad \text{by } (*) \\ &\in \mathbb{Z}, && \text{since } f(k-1), f(k) \in \mathbb{Z} \text{ by (I.A.)} \end{aligned}$$

$\therefore$ $P(k+1)$ holds, if $P(k-1), P(k)$ hold.

(conclusion) Hence, if $x + 1/x \in \mathbb{Z}$, we have proved:

$$\begin{aligned} &P(1) \text{ and } P(2) \text{ hold, and} \\ &P(k-1) \text{ and } P(k) \implies P(k+1), \text{for } k \geqslant 2, \end{aligned}$$

so that, by PMI, $P(n)$ holds for all $n \in \mathbb{N}$, i.e.
if $x + 1/x \in \mathbb{Z}$ then $x^n + 1/x^n \in \mathbb{Z}$ for all $n \in \mathbb{N}$. □

Remark 3.1.6. In the example above, we defined $f(n)$. We did this since, with the relational operator of $P(n)$ being "$\in$", "LHS of P(n)" is somewhat unnatural, and $f(n)$ is, in any case, much shorter.

3.2 Variants of induction

We have already seen two variants of induction, in the two examples of this chapter. The most common variant, **simple induction** was what was actually given in Definition 3.1.1.

As suggested in Note 3.1.2, we are now going to have a little fun, revisiting the ladder concept, and imagining beasts with various afflictions that need to be catered for, to customise other forms of induction, and thereby show those beasts can still climb the ladder. For each variant, we give a bit of a fun "back story" to explain any modifications to the *base case*, *inductive step*, and the *conclusion*:

Simple induction. This is just the usual induction, with:
(base case) Show $P(1)$ holds.
(inductive step) Show $P(k) \implies P(k+1)$, for general k.
(conclusion) $P(n)$ holds for all $n \in \mathbb{N}$.

Simple induction – Aladdin's version.
Here, the first rungs of the ladder are broken, but Aladdin can use a wish to get to rung n_0:
(base case) Show $P(n_0)$ holds.
(inductive step) Show $P(k) \implies P(k+1)$, for general $k \geqslant n_0$.
(conclusion) $P(n)$ holds for all $n \in \mathbb{N}$, such that $n \geqslant n_0$.

Secondary induction – giant version.
The giant can't bend his knees and can only do two rungs at a time.
If he can only start by getting to the first rung, then he is only able to get to odd rungs.
So he must be able to start by getting to both rungs 1 and 2:
(base case) Show $P(1)$ and $P(2)$ hold.
(inductive step) Show $P(k) \implies P(k+2)$, for general k.
(conclusion) $P(n)$ holds for all $n \in \mathbb{N}$.

Secondary induction – Jake the Peg version.
Jake the Peg is a 3-legged man. (A variant of this one was used in Example 3.1.5.)
To climb the ladder Jake needs to have 2 feet on the 2 previous rungs,
to push up with his 3rd leg to get on the next rung.
To start he needs to be able to somehow get on rungs 1 and 2:
(base case) Show $P(1)$ and $P(2)$ hold.
(inductive step) Show $P(k), P(k+1) \implies P(k+2)$, for general k.
(conclusion) $P(n)$ holds for all $n \in \mathbb{N}$.

Complete[*] induction – polypus version.
The polypus is a strange animal. As it climbs new legs spring into existence.
To climb the ladder it needs to have k feet on the k previous rungs
to push up with its newly sprung $(k+1)^{\text{st}}$ leg to get on the next rung.
At the start it only has 2 legs. So to start, it only needs to be able to get on the first rung:
(base case) Show $P(1)$ holds.
(inductive step) Show $P(1), P(2), \ldots, P(k) \implies P(k+1)$, for general k.
(conclusion) $P(n)$ holds for all $n \in \mathbb{N}$.

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[*] *Complete induction* is also called **strong induction**, though *strong induction* often refers to any induction "stronger" than *simple induction*.

Exercise Set 3.

1. Prove for any natural number n that
   (a) $1+3+5+\cdots+2n-1=n^2$;
   (b) $1^2+2^2+3^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$;
   (c) $1^3+2^3+3^3+\cdots+n^3=\frac{1}{4}n^2(n+1)^2$;
   (d) $1^2+4^2+7^2+\cdots+(3n-2)^2=\frac{1}{2}n(6n^2-3n-1)$;
   (e) $2^2+5^2+8^2+\cdots+(3n-1)^2=\frac{1}{2}n(6n^2+3n-1)$.

2. Prove that for any natural number n,
$$2(\sqrt{n+1}-1)<1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}<2\sqrt{n}.$$

3. Prove $3^n>2^n$ for all natural numbers n.

4. Prove *Bernoulli's Inequality* which states:

   If $x\geqslant -1$ then $(1+x)^n\geqslant 1+nx$ for all natural numbers n.

5. Prove that for any natural number $n\geqslant 2$,
$$\left(1-\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{\sqrt{3}}\right)\cdots\left(1-\frac{1}{\sqrt{n}}\right)<\frac{2}{n^2}.$$

6. Prove that for any natural number n,
$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\leqslant\frac{1}{\sqrt{3n+1}}.$$

7. Prove that $7^{2n}-48n-1$ is divisible by 2304 for every natural number n.

8. For every natural number n, show that
$$u_n=\frac{(1+\sqrt{5})^n-(1-\sqrt{5})^n}{2^n.\sqrt{5}}$$
   is a natural number.
   In fact, u_n is the nth Fibonacci number.

9. Prove that for all $n\in\mathbb{N}$ and $x\in\mathbb{Z}\setminus\{1\}$, x^n-1 is divisible by $x-1$.

10. Given that $a_1=-2$, $a_2=-16$ and $a_{n+2}=8a_{n+1}-15a_n$, prove $a_n=3^n-5^n$ for all $n\in\mathbb{N}$.

11. Consider all possible subsets of the set $\{1,2,\ldots,n\}$ which do not contain any consecutive numbers.
    Prove that the sum of squares of the products of the numbers in these sets is $(n+1)!-1$.

12. Use Mathematical Induction to prove the following propositions, for $n \in \mathbb{N}$ (unless further restricted).
    (a) $2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 2$.
    (b) $\dfrac{n^3}{3} + \dfrac{n^5}{5} + \dfrac{7n}{15}$ is an integer.
    (c) For all $n > 2$, the sum of the interior angles of a convex polygon of n sides is $180(n-2)^\circ$.
    (d) If a set A contains n elements then the power set of A contains 2^n elements.
    (e) The Fibonacci numbers are defined by:
    $$F_1 = F_2 = 1, \text{ and } F_{n+2} = F_n + F_{n+1}, n \geqslant 1.$$
    Show that
    $$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1.$$
    (f) $1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{n}{3}(2n-1)(2n+1)$.
    (g) $2^n \geqslant n^2$ for all $n \geqslant 4$.
    (h) $2^n \geqslant n^5$ for all $n \geqslant 23$.
    (i) $e^n \geqslant 10n$ for all $n \geqslant 4$.
    (j) $\dfrac{1}{1 \times 3} + \dfrac{1}{3 \times 5} + \cdots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$.

13. Prove that
$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 1$$
for all integers $n > 1$.

The lesson with this example is that sometimes one needs to prove something stronger. A naive (perhaps, that should be: *first*) attempt at the problem, might be to define
$$P(n) : \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 1$$
and try to prove $P(n)$ by induction for all $n \geqslant 2$, but at the inductive step we seem to be stuck because we want to increase the righthand side.
So define the RHS of $P(n)$ to be slightly smaller than 1 such that it depends on n.

14. (AMO 2005, Problem 4) Show that, for $n \in \mathbb{N}$, there exists an $x \in \mathbb{N}$ such that
$$\sqrt{x + 2004^n} + \sqrt{x} = (\sqrt{2005} + 1)^n.$$

15. Given
$$x_1 = 1$$
$$x_{n+1} = x_n + \frac{1}{2x_n}, n \geqslant 1,$$
prove that $\lfloor 25x_{625} \rfloor = 625$.

16. (You may like to come back to this problem after having done some Number Theory.) What are the last two digits of 2^{222}?

    Some ideas: You might be wondering what Euler's Theorem or Fermat's Little Theorem have to say about this. You should be able to see that the problem is equivalent to asking what 2^{222} *modulo* 100. Since 100 is not prime, we cannot use Fermat's Little Theorem, and we cannot use Euler's Theorem directly since 2 and 100 are not coprime, but one could find what 2^{222} *modulo* 25 by Euler's Theorem and lift that to a result *modulo* 100.

    Another approach is repeated squaring. This certainly works, and on a computer is quite fast, but for a human it's a bit tedious.

    Yet another approach is to recover something that looks a little like a Fermat's Little Theorem result, namely finding a cycle such that:

    $$2^{k+\ell} \equiv 2^k \pmod{100}.$$

    In fact, one finds that

    $$2^2 \equiv 4 \pmod{100} \quad \text{and} \quad 2^{12} = 4096 \equiv -4 \pmod{100}.$$

    Note that this does *not* imply that 2^{10} is congruent to -1 *modulo* 100, since 2^2 does *not* have a multiplicative inverse *modulo* 100, but nevertheless one can show (by induction):

    $$2^{2+10n} \equiv (-1)^n 2^2 \pmod{100}$$

    and use it to solve the problem.

17. Prove any natural number $n \geqslant 2$ is the product of primes.

    Hint. Define a $P(n)$ and use an Aladdin's version of *complete induction* where the *base case* proves $P(2)$ holds.

18. Prove that with 3c and 5c stamps, any postage denomination greater than 7c is possible.

    Hint. Define a $P(n)$ and use a variant of the *giant's version,* where the *base case* proves $P(8)$, $P(9)$ and $P(10)$ hold, and the *inductive step* proves $P(k) \implies P(k+3)$.

19. Prove $10 \,|\, (n^5 - n)$ for all nonnegative integers n.

    Hint. Define a $P(n)$ and as per the traditional *giant's version,* prove $P(k) \implies P(k+2)$. What's the *base case?*

20. Prove an equilateral triangle can be partitioned into n subtriangles that are all equilateral, for all $n \geqslant 6$.

    Hint. Define a $P(n)$. Prove $P(4)$, $P(6)$, $P(7)$, $P(8)$. Prove $P(k) \implies P(k+3)$, using $P(4)$, as per a variant of the *giant's version.* What's the *base case?*