CHAPTER 11

Algebra: Inequalities

11.1 Introduction

At school when you met the topic of *Inequalities* you were interested in finding the set of solutions for which a given inequality is satisfied, e.g. you might be asked:

For which $x \in \mathbb{R}$, is $x^2 \geqslant 3x - 2$.

One technique for *solving* such an inequality is rearrange it to have right hand side 0, and factorise the resulting left hand quadratic. It's then straightforward to determine the sign of that left hand side expression, and hence find the solution of the inequality as interval(s) of $\mathbb{R}$:

$$\begin{aligned} x^2 &\geqslant 3x - 2 \\ x^2 - 3x + 2 &\geqslant 0 \\ (x-2)(x-1) &\geqslant 0 \end{aligned}$$

Now, we observe that $(x-2)(x-1)$ is *positive* if both factors are *negative* or both factors are *positive*, and is *zero* when either factor is *zero*, i.e. the solution is

$$x \leqslant 1 \text{ or } x \geqslant 2.$$

Of course, one needs to start this way to gain some familiarity with how *inequations* differ from *equations*.

However, our interest in these lectures is to prove certain *Inequalities* hold for all $x \in \mathbb{R}$. One technique for this might be to *solve* an inequality as above, and show that the solution interval is all of $\mathbb{R}$, but for the sorts of inequalities with which we will consider, often involving several variables, this is generally not a useful approach. Instead, we will build up an armoury of *Standard Inequalities* and use these to prove the results we are after. Before we do that, let's start near the beginning.

11.2 Symbols and Elementary Rules

No doubt, you are very familiar with the symbols

$$> \quad \geqslant \quad < \quad \leqslant$$

but you probably have not thought much about the rules they obey. Let us start with some properties of *real* numbers.

- A real number can only be one of *positive*, *negative* or 0. Put another way, for a real number r, one of r or $-r$ is *positive* or else $r = 0$.
- The sum or product of two *positive* numbers is *positive*.
- Of course, for any real number r, $r + 0 = r$ and $r \cdot 0 = 0$.

Now, recognise that $a > b$ *means* that $a - b$ is *positive*. Also $a \geqslant b$ *means* that *either* $a > b$ *or* $a = b$. (Sometimes, it is useful to interpret $a = b$ as: $a - b$ is 0.) Of course, $a < b$ *means* $b > a$; and $a \leqslant b$ *means* $b \geqslant a$.

So now let's look at some rules that involve $>$ and $\geqslant$ (and $<$ and $\leqslant$). In each rule a, b, c, d are real numbers. The proofs will seem obvious – notice in each case we have used just *real* number properties (the main ones we use are mentioned above.)

- If $a > b$ then $a + c > b + c$. *(Note that c is allowed to be negative.)*

  Proof. Let $a > b$, i.e. $a - b$ is *positive*. Now $a - b = (a + c) - (b + c)$. So $(a + c) - (b + c)$ is *positive*, i.e. $a + c > b + c$. □

- If $a > b$ and c is *positive* then $ac > bc$.

  Proof. Let $a > b$, i.e. $a - b$ is *positive*. Also, let c be *positive*. Thus, $(a - b)c = ac - bc$ is *positive*, i.e. $ac > bc$. □

- If $a > b$ and c is *negative* then $ac < bc$.

  Proof. Let $a > b$ and c be *negative*, i.e. $a - b$ and $-c$ are *positive*. Thus, $(a - b)(-c) = bc - ac$ is *positive*, i.e. $bc > ac$ (or equivalently $ac < bc$). □

- Always $a^2 \geqslant 0$. (The *minimum value* property of a square.)

  Proof. If a is *positive* then $a \cdot a = a^2$ is *positive*. If $-a$ is *positive* then $(-a) \cdot (-a) = a^2$ is *positive*. If a is 0 then $a \cdot a = a^2$ is 0. Hence a^2 is *positive* or 0, i.e. $a^2 \geqslant 0$. □

- If $a > b$ and $b > c$ then $a > c$. *(Transitivity property)*

  Proof. Let $a > b$ and $b > c$, i.e. $a - b$ and $b - c$ are *positive*. Hence $(a - b) + (b - c)$ is *positive*. But $(a - b) + (b - c) = a - c$. Hence $a - c$ is *positive*, i.e. $a > c$. □

- If $a > b$ and $c > d$ then $a + c > b + d$.

  Proof. Let $a > b$ and $c > d$, i.e. $a - b$ and $c - d$ are *positive*. Hence $(a - b) + (c - d)$ is *positive*. But $(a - b) + (c - d) = (a + c) - (b + d)$. Hence $(a + c) - (b + d)$ is *positive*, i.e. $a + c > b + d$. □

- If $0 < a < b$ then $\frac{1}{a} > \frac{1}{b} > 0$.

  Proof. *Exercise.* □

- If $0 < a < 1$ and n is a natural number then $0 < a^n < 1$.

  Proof. *Exercise. (Hint: use Mathematical Induction.)* □

Observe that if we let $a = x/y$, $b = 1$ and $c = y$ then the second rule becomes:

If $\frac{x}{y} > 1$ and y is *positive* then $x > y$.

Thus, we may prove that $x > y$ by showing *either*

- $x - y$ is *positive; or*
- $\frac{x}{y} > 1$ provided that y is *positive.*

Example 11.2.1. *(i) If x, y are* distinct positive *numbers then*

$$x^3 + y^3 > x^2y + xy^2.$$

Proof. We will show that $(x^3 + y^3) - (x^2y + xy^2)$ is *positive*. Now

$$\begin{aligned}(x^3 + y^3) - (x^2y + xy^2) = x^3 - x^2y + y^3 - xy^2 &= x^2(x - y) + y^2(y - x)\\ &= (x^2 - y^2)(x - y)\\ &= (x + y)(x - y)^2.\end{aligned}$$

Now, by our properties of *real* numbers and our rules, both $x + y$ and $(x - y)^2$ are *positive*, and hence their product is *positive*, i.e. $x^3 + y^3 > x^2y + xy^2$. □

(ii) If $x > y > 0$ then

$$4x^3(x - y) > x^4 - y^4.$$

Proof. Since $x > y > 0$ we have $x > 0$ (using the *transitivity property*). Now $x^4 - y^4 = (x - y)(x + y)(x^2 + y^2)$ and each of $x - y$, $x + y$ and $x^2 + y^2$ is *positive*. (Check the details!) Hence $x^4 - y^4$ is *positive*. We are now in a position to prove the result by showing that

$$\frac{4x^3(x - y)}{x^4 - y^4} > 1.$$

But,

$$\begin{aligned}\frac{4x^3(x - y)}{x^4 - y^4} &= \frac{4x^3(x - y)}{(x - y)(x^3 + x^2y + xy^2 + y^3)}\\ &= \frac{4x^3}{x^3 + x^2y + xy^2 + y^3} \quad \text{since } x - y \neq 0\\ &= \frac{4}{1 + \frac{y}{x} + \frac{y^2}{x^2} + \frac{y^3}{x^3}} \quad \text{since } x \neq 0\\ &> 1\end{aligned}$$

The last step is valid since $0 < \frac{y}{x} < 1$. (Check all the skipped details!)
Thus, we may deduce that $4x^3(x - y) > x^4 - y^4$. □

11.3 Absolute values

Absolute values are often most easily treated from a geometric point of view. In particular, *the absolute value of a number measures its distance from* 0. We can extend this idea to interpret

$$|x - a|$$

as *the distance of x from a*. Thus to solve

$$|x + 1| < 3$$

we may first rewrite it as

$$|x - (-1)| < 3$$

and interpret it as: *the distance of x from -1 is less than 3* giving us $-4 < x < 2$. (To see this, draw a number line.)

Algebraically, we have the following definition for $|x|$,

$$|x| = \begin{cases} x, & \text{if } x \geqslant 0,\\ -x, & \text{if } x < 0.\end{cases}$$

and note that for a positive real number a we have that

$$|x| < a \quad \text{if and only if} \quad -a < x < a.$$

To gain some familiarity with manipulating absolute values, try the following exercises.

Exercises – absolute values.

1. Find the solution interval(s) for the following inequalities.

(i) $\lvert x+7\rvert > 3$	(iii) $\lvert x-2\rvert \geqslant \lvert 2x+3\rvert$
(ii) $\lvert 2x-7\rvert < 2$	(iv) $1-x \geqslant \lvert x-1\rvert$

11.4 Triangle Inequality

The name of this inequality comes from the geometric observation that the length of a side of triangle must lie between the difference and sum of the other two sides:

Theorem 11.4.1 (Triangle Inequality).

$$\big||x|-|y|\big| \leqslant |x+y| \leqslant |x|+|y|$$

for any real numbers x, y.

11.5 Squares are never negative

We identified this property earlier, but it's so important it bears repeating and putting it in its own section.

The square of a real number is never negative, i.e.

$$x^2 \geqslant 0, \quad \text{with equality} \iff x = 0,$$

or more generally

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geqslant 0, \quad \text{with equality} \iff x_1 = x_2 = \cdots = x_n = 0.$$

Exercises – squares are non-negative.

2. Prove that for any non-negative a, b,

$$\frac{a+b}{2} \geqslant \sqrt{ab}.$$

This result is AM-GM (which we will discuss further later) for the case $n = 2$.

3. For arbitrary $a, b, c \in \mathbb{R}$, prove $a^2 + b^2 + c^2 \geqslant ab + bc + ca$.

4. (1990 USSR MO Q1) Prove that for arbitrary $t \in \mathbb{R}$, the inequality $t^4 - t + \frac{1}{2} > 0$ holds.

5. Let $a, b, c, d \in \mathbb{R}$. Prove that the numbers $a - b^2$, $b - c^2$, $c - d^2$, $d - a^2$ cannot all be larger than $\frac{1}{4}$.

6. Prove that $(a + 5b)(3a + 2b) \geqslant (a + 9b)(2a + b)$ for all $a, b \in \mathbb{R}$.

7. Prove $(p + 2)(q + 2)(p + q) \geqslant 16pq$ for all $p, q \geqslant 0$.

8. Prove that $a^2(1 + b^4) + b^2(1 + a^4) \leqslant (1 + a^4)(1 + b^4)$ for all $a, b \in \mathbb{R}$.

9. If $x, y, z \in \mathbb{R}$ such that $x + y + z = 1$, prove that $x^2 + y^2 + z^2 \geqslant \frac{1}{3}$.

11.6 Arithmetic, Geometric and Harmonic Means

For a positive real number sequence $x_1, x_2, \ldots, x_n$, these means are defined by

$$\text{The \textbf{Arithmetic Mean} (AM)} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

$$\text{The \textbf{Geometric Mean} (GM)} = \sqrt[n]{x_1 x_2 \cdots x_n}$$

$$\text{The \textbf{Harmonic Mean} (HM)} = \left(\frac{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}{n}\right)^{-1} = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}.$$

Theorem (AM-GM-HM). *Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Then*

$$\begin{aligned} &\mathrm{AM}(x_1, x_2, \ldots, x_n) \geqslant \mathrm{GM}(x_1, x_2, \ldots, x_n) \geqslant \mathrm{HM}(x_1, x_2, \ldots, x_n) \\ \textit{i.e.}\quad &\frac{x_1 + x_2 + \cdots + x_n}{n} \geqslant \sqrt[n]{x_1 x_2 \cdots x_n} \geqslant \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \end{aligned}$$

with equality $\iff x_1 = x_2 = \cdots = x_n$.

Proof. The statement can be proved by induction. The case $n = 1$ is trivially true. The case $n = 2$ follows after starting with

$$(\sqrt{x_1} - \sqrt{x_2})^2 \geqslant 0.$$

This gives $\mathrm{AM}(x_1, x_2) \geqslant \mathrm{GM}(x_1, x_2)$, from which can be deduced $\mathrm{GM}(x_1, x_2) \geqslant \mathrm{HM}(x_1, x_2)$.

Following Cauchy's approach, we deduce the AM-GM inequality for $n = 2k$ from the cases $n = 2$ and $n = k$. Similarly, we deduce the GM-HM inequality for $n = 2k$ from the cases $n = 2$ and $n = k$.

At this stage, one has AM-GM-HM for all powers of 2. To deduce for general n, first let $\alpha = \mathrm{AM}(x_1, x_2, \ldots, x_n)$. Then add in $(m - n)$ extra αs, where m is a power of 2. Then

$$\begin{aligned} \alpha = \mathrm{AM}(x_1, x_2, \ldots, x_n) &= \mathrm{AM}(x_1, x_2, \ldots, x_n, \alpha, \ldots, \alpha) \\ &\geqslant \mathrm{GM}(x_1, x_2, \ldots, x_n, \alpha, \ldots, \alpha) \\ &= \sqrt[m]{x_1 x_2 \cdots x_n \alpha^{m-n}} \\ \alpha^m &\geqslant x_1 x_2 \cdots x_n \alpha^{m-n} \\ \alpha^n &\geqslant x_1 x_2 \cdots x_n \\ \mathrm{AM}(x_1, x_2, \ldots, x_n) = \alpha &\geqslant \sqrt[n]{x_1 x_2 \cdots x_n} = \mathrm{GM}(x_1, x_2, \ldots, x_n) \end{aligned}$$

The proof of the GM-HM inequality for general n is similar. Start with $\alpha = \mathrm{HM}(x_1, x_2, \ldots, x_n)$, again add in $(m - n)$ extra αs, and deduce that $\mathrm{HM}(x_1, x_2, \ldots, x_n) \leqslant \mathrm{GM}(x_1, x_2, \ldots, x_n)$. ☐

Exercises – AM-GM Examples.

10. (1995 AIC* Q3) If $1 \leqslant n \in \mathbb{Z}$, prove that $(n + 1)^n \geqslant 2^n n!$. When does equality hold?

*AIC (Australian Intermediate Contest) was a 5-question fore-runner of the AIMO.

11. Prove that $(a+b)(b+c)(c+a) \geqslant 8abc$ for all nonnegative $a, b, c \in \mathbb{R}$.
12. Prove that $a^2+b^2+c^2 \geqslant ab+bc+ca$ for all $a, b, c \in \mathbb{R}$.
13. Prove that $x(a-x) \leqslant a^2/4$ if $a, x \in \mathbb{R}$, $x > 0$.
14. Prove that $a + 1/a \geqslant 2$, for all positive $a \in \mathbb{R}$.
15. (1961 Swedish MO Q2) For all positive $x_1, x_2, \dots, x_n \in \mathbb{R}$, prove that
$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_n}{x_1} \geqslant n.$$
16. If $0 < a, b, c, d \in \mathbb{R}$ such that $a+b+c+d=1$, prove that
$$\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} + \sqrt{4d+1} < 6.$$

Exercises – AM-HM Examples.

17. (1998 Irish MO Q7) Prove that if $0 < a, b, c \in \mathbb{R}$ then
$$\frac{9}{a+b+c} \leqslant \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} \leqslant \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$
18. (1976 British MO Q2) Prove that if $0 < a, b, c \in \mathbb{R}$ then
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geqslant \frac{3}{2}.$$
19. For positive $x_1, x_2, x_3, x_4 \in \mathbb{R}$, prove that
$$\frac{x_1+x_3}{x_1+x_2} + \frac{x_2+x_4}{x_2+x_3} + \frac{x_3+x_1}{x_3+x_4} + \frac{x_4+x_2}{x_4+x_1} \geqslant 4.$$

11.7 The Cauchy-Schwarz Inequality

Theorem 11.7.1 (Cauchy-Schwarz Inequality). *For $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in \mathbb{R}$,*
$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geqslant (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2$$
with equality if and only if
$$a_1 : b_1 = a_2 : b_2 = \cdots = a_n : b_n.$$
The Cauchy-Schwarz Inequality is most easily remembered in terms of vectors:
$$\|\underset{\sim}{\boldsymbol{a}}\|^2 \|\underset{\sim}{\boldsymbol{b}}\|^2 \geqslant |\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}}|^2$$
$$\text{i.e.} \quad \sum_i a_i^2 \cdot \sum_i b_i^2 \geqslant \Big(\sum_i a_i b_i\Big)^2$$
where the $a_i, b_i \in \mathbb{R}$ for all i, with equality if and only if $\underset{\sim}{\boldsymbol{a}} \parallel \underset{\sim}{\boldsymbol{b}}$.

Proof. Firstly, we give a proof without using vectors.
Since $(a_ix+b_i)^2 \geqslant 0$,

$$\sum_{i=1}^{n}(a_ix+b_i)^2 \geqslant 0$$
$$\Big(\sum_{i=1}^{n} a_i^2\Big)x^2 + 2\Big(\sum_{i=1}^{n} a_ib_i\Big)x + \Big(\sum_{i=1}^{n} b_i^2\Big) \geqslant 0$$

The lefthand side of the last inequality is a quadratic polynomial in x. Since it is nonnegative its graph either touches the x-axis at one point (i.e. the polynomial has *exactly* one zero) or is entirely above the x-axis (i.e. the polynomial has no real zeros). Consequently, polynomial's discriminant is *non-positive:*

$$4\Big(\sum_{i=1}^{n} a_ib_i\Big)^2 - 4\Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big) \leqslant 0$$
$$\Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big) \geqslant \Big(\sum_{i=1}^{n} a_ib_i\Big)^2.$$

Equality occurs when the polynomial has *exactly* one zero, which is to say that there is an x such that $a_ix+b_i=0$ for all i, which is equivalent to saying the ratios $a_i : b_i$ are equal for all i.

In terms of vectors, we have the identity

$$\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}} = \|\underset{\sim}{\boldsymbol{a}}\|\,\|\underset{\sim}{\boldsymbol{b}}\|\cos\theta,$$

where θ is the angle between the 'tails' of the vectors. Squaring and using $|\cos\theta| \leqslant 1$, we have

$$\|\underset{\sim}{\boldsymbol{a}}\|^2\|\underset{\sim}{\boldsymbol{b}}\|^2 \geqslant \|\underset{\sim}{\boldsymbol{a}}\|^2\|\underset{\sim}{\boldsymbol{b}}\|^2\cos^2\theta$$
$$= |\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}}|^2.$$

Equality occurs if and only if

$$\cos\theta = 1 \iff \underset{\sim}{\boldsymbol{a}} \parallel \underset{\sim}{\boldsymbol{b}}.$$ □

Exercises – Cauchy-Schwarz.

20. Prove that for $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and positive $h_1, h_2, \ldots, h_n \in \mathbb{R}$, where $n \in \mathbb{N}$,

$$\sum_{i=1}^{n}\frac{a_i^2}{h_i} \geqslant \frac{(\sum_{i=1}^{n} a_i)^2}{\sum_{i=1}^{n} h_i}.$$

21. If $0 < a, b, c, d \in \mathbb{R}$, prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{4}{c}+\frac{16}{d} \geqslant \frac{64}{a+b+c+d}.$$

22. For all $a, b, c \in \mathbb{R}$, prove that $a^2+b^2+c^2 \geqslant ab+bc+ca$.

23. If $0 < a, b, c, d \in \mathbb{R}$ such that $(a^2 + b^2)^3 = c^2 + d^2$, prove that

$$\frac{a^3}{c} + \frac{b^3}{d} \geqslant 1.$$

24. (1990 USSR MO Q10) If $0 < a_1, a_2, \ldots, a_n \in \mathbb{R}$ such that $a_1 + a_2 + \cdots + a_n = 1$, prove

$$\frac{a_1^2}{a_1 + a_2} + \frac{a_2^2}{a_2 + a_3} + \cdots + \frac{a_n^2}{a_n + a_1} \geqslant \frac{1}{2}.$$

11.8 Rearrangements

Theorem 11.8.1 (Rearrangement Inequality). *Let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in \mathbb{R}$ such that $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ and $y_1 \leqslant y_2 \leqslant \cdots \leqslant y_n$, where $n \in \mathbb{N}$ and let $z_1, z_2, \ldots, z_n$ be any permutation (rearrangement) of $y_1, y_2, \ldots, y_n$. Then*

$$x_1y_n + x_2y_{n-1} + \cdots + x_ny_1 \leqslant x_1z_1 + x_2z_2 + \cdots + x_nz_n \leqslant x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

Remark. Suppose we have two number sequences (terms x_i and y_j, respectively) of length n. Then the Rearrangement Inequality says that of the expressions one can form that are sums of n product pairs x_iy_j, the *minimum* value is achieved when the largest x_i is paired with the smallest y_j, the second-largest x_i is paired with the second-smallest y_j, and so on; and the *maximum* value is achieved when the largest x_i is paired with the largest y_j, the second-largest x_i is paired with the second-largest y_j, etc. Any other choice of pairings gives a value that lies between these minimum and maximum values.

Partial proof of Rearrangement Inequality. The following is a 'start' giving the general idea. Suppose $x_1 \leqslant x_2 \leqslant x_3$ and $y_1 \leqslant y_2 \leqslant y_3$. Then

$$\begin{aligned}
(x_3 - x_2)(y_3 - y_2) &\geqslant 0 \\
x_2y_2 + x_3y_3 &\geqslant x_2y_3 + x_3y_2 \\
x_1y_1 + x_2y_2 + x_3y_3 &\geqslant x_1y_1 + x_2y_3 + x_3y_2
\end{aligned}$$

Proceeding in this way leads to a general proof. □

Exercises – rearrangements.

25. For all $a, b, c \in \mathbb{R}$ prove $a^2 + b^2 + c^2 \geqslant ab + bc + ac$.

26. (1935 Eötvös Competition Q1) Let $y_1, y_2, \ldots, y_n$ be any permutation of the positive real numbers $x_1, x_2, \ldots, x_n$. Prove that

$$\frac{x_1}{y_1} + \frac{x_2}{y_2} + \cdots + \frac{x_n}{y_n} \geqslant n.$$

27. (1976 British MO Q2) For positive $a, b, c \in \mathbb{R}$, prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geqslant \frac{3}{2}.$$

28. (2002 Canadian MO Q3) For positive $x, y, z \in \mathbb{R}$, prove that

$$\frac{x^3}{yz} + \frac{y^3}{zx} + \frac{z^3}{xy} \geqslant x + y + z.$$

11.9 Optimisation applications

Later you will probably use calculus almost exclusively, when you need to find the maximum or minimum of a function, but you shouldn't forget that you can often use inequalities techniques for this purpose, and such solutions are often exquisitely short and *elegant!*

Example 11.9.1. *(i) Find the minimum value of $x^2+8x+23$ and the value(s) of x for which this minimum is attained.*

Solution. *We complete the square:*

$$\begin{aligned} x^2+8x+23 &= (x+4)^2+7 \\ &\geqslant 7, \qquad \text{since the square } (x+4)^2 \geqslant 0. \end{aligned}$$

Thus the expression is bounded below by $7 = 0+7$, and since at $x=-4$ we have $(x+4)^2=0$, in fact the lower bound is attained, i.e. the expression has a minimum *value 7 that is attained at $x=-4$.*

(ii) (Adapted from AIMO 2008 Q10) Find the maximum value of E satisfying

$$A+B+C+D+E=0 \tag{11.9.1}$$

$$A^2+B^2+C^2+D^2+E^2=80. \tag{11.9.2}$$

Solution. *By AM-GM, for $n=2$, we have*

$$\frac{A^2+B^2}{2} \geqslant AB,\ \frac{A^2+C^2}{2} \geqslant AC,\ \dots,\ \frac{C^2+D^2}{2} \geqslant CD,$$

with these all becoming equalities *if $A=B=C=D$. We will use this in step (11.9.4) below. Isolating E in (11.9.1) we have*

$$E = -(A+B+C+D) \tag{11.9.3}$$

$$\begin{aligned} E^2 &= (A+B+C+D)^2 \\ &= A^2+B^2+C^2+D^2+2AB+2AC+\cdots+2CD \\ &\leqslant A^2+B^2+C^2+D^2+(A^2+B^2)+(A^2+C^2)+\cdots+(C^2+D^2) \qquad (11.9.4) \\ &= 4(A^2+B^2+C^2+D^2), \qquad \text{since from } 2AB,\ 2AC \text{ and } 2AD \text{ we obtain } 3A^2 \text{ and by symmetry there are as many } A^2\text{s as } B^2\text{s, } C^2\text{s and } D^2\text{s} \\ &= 4(80-E^2), \qquad \text{using (11.9.2)} \\ \therefore 5E^2 &\leqslant 4\cdot 80 \\ E^2 &\leqslant 4\cdot 16 \\ E &\leqslant 8, \qquad \text{with equality if } A=B=C=D\ (= -E/4 \text{ by (11.9.3)}) \end{aligned}$$

Therefore, the maximum value of E is 8, attained when $A=B=C=D=-2$.

Later, in your mathematics career you will learn how to do the following problem by *Lagrange multipliers*, but AM-GM is quicker!

Exercises – Optimisation Application of AM-GM.

29. Given $a,b,c>0$, find the minimum value of $a+2b+7c$ such that $a^2b^5c=1$.

    Hint. Let $x_1 = x_2 = \frac{a}{2}$, $x_3 = \cdots = x_7 = \frac{2b}{5}$, and $x_8 = 7c$, and note that minimum value is a nasty surd, but it's not difficult to obtain. We want the exact expression, but once you've found it you can find an approximate value with a calculator if you like `;-)`.

11.10 Generalising AM-GM-HM

The following theorem generalises the AM-GM-HM Theorem. Firstly, the **Quadratic Mean** (QM) is the 2-Power Mean:

$$\mathrm{QM}(x_1, x_2, \ldots, x_n) = \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}.$$

In general, the k-**Power Mean** (PM_k), $k \in \mathbb{Z}$, is given by

$$\mathrm{PM}_k(x_1, x_2, \ldots, x_n) = \begin{cases} \sqrt[k]{\dfrac{x_1^k + x_2^k + \cdots + x_n^k}{n}}, & \text{if } k \neq 0 \\ \sqrt[n]{x_1 \cdot x_2 \cdots x_n}, & \text{if } k = 0. \end{cases}$$

With this definition, the AM is the 1-Power Mean, the GM is the 0-Power Mean, and the HM is the -1-Power Mean.

Theorem 11.10.1 (Power Mean (Hölder Mean)). *Let* $0 < x_1, x_2, \ldots, x_n \in \mathbb{R}$. *Then*

$$k \geqslant \ell \implies \mathrm{PM}_k(x_1, x_2, \ldots, x_n) \geqslant \mathrm{PM}_\ell(x_1, x_2, \ldots, x_n),$$

with equality $\iff x_1 = x_2 = \cdots = x_n$.

In particular,

$$\mathrm{QM}(x_1, \ldots, x_n) \geqslant \mathrm{AM}(x_1, \ldots, x_n) \geqslant \mathrm{GM}(x_1, \ldots, x_n) \geqslant \mathrm{HM}(x_1, \ldots, x_n).$$

Exercises – QM-AM-HM and Power Mean.

30. Prove the QM-AM part of the above theorem for the case $n = 2$.

31. Show that, if $a, b > 0$ and $a + b = 1$ then $\quad \left(a + \dfrac{1}{a}\right)^2 + \left(b + \dfrac{1}{b}\right)^2 \geqslant \dfrac{25}{2}.$

32. (1976 Vietnam MO B3 Q6) For positive $x_1, x_2, \ldots, x_n \in \mathbb{R}$ such that $x_1 + x_2 + \cdots + x_n = 1$ and nonnegative $k \in \mathbb{Z}$, prove that

$$\frac{1}{x_1^k} + \frac{1}{x_2^k} + \cdots + \frac{1}{x_n^k} \geqslant n^{k+1}.$$

11.11 The Chebyshev Inequality

The following theorem essentially extends the Rearrangement Inequality, and we show that it follows from the Rearrangement Inequality.

Theorem 11.11.1 (Chebyshev Inequality). *If* $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ *and* $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$ *then*

$$\frac{a_1b_1 + a_2b_2 + \cdots + a_nb_n}{n} \geqslant \frac{(a_1 + a_2 + \cdots + a_n)}{n} \cdot \frac{(b_1 + b_2 + \cdots + b_n)}{n} \geqslant \frac{a_1b_n + a_2b_{n-1} + \cdots + a_nb_1}{n}.$$

Proof. Assume $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ and $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$. Then by the Rearrangement Inequality, we have the following n inequalities:

$$\begin{aligned}
a_1b_1 + a_2b_2 + \cdots + a_nb_n &= a_1b_1 + a_2b_2 + \cdots + a_nb_n &&\geqslant a_1b_n + a_2b_{n-1} + \cdots + a_nb_1\\
a_1b_1 + a_2b_2 + \cdots + a_nb_n &\geqslant a_1b_2 + a_2b_3 + \cdots + a_nb_1 &&\geqslant a_1b_n + a_2b_{n-1} + \cdots + a_nb_1\\
a_1b_1 + a_2b_2 + \cdots + a_nb_n &\geqslant a_1b_3 + a_2b_4 + \cdots + a_nb_2 &&\geqslant a_1b_n + a_2b_{n-1} + \cdots + a_nb_1\\
&\vdots\\
a_1b_1 + a_2b_2 + \cdots + a_nb_n &\geqslant a_1b_n + a_2b_1 + \cdots + a_nb_{n-1} &&\geqslant a_1b_n + a_2b_{n-1} + \cdots + a_nb_1.
\end{aligned}$$

Now, adding these n inequalities, followed by dividing through by n^2 gives the result:

$$\begin{aligned}
n(a_1b_1 + a_2b_2 + \cdots + a_nb_n) &\geqslant (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n)\\
&\geqslant n(a_1b_n + a_2b_{n-1} + \cdots + a_nb_1)\\
\therefore \frac{a_1b_1 + a_2b_2 + \cdots + a_nb_n}{n} &\geqslant \frac{(a_1 + a_2 + \cdots + a_n)}{n} \cdot \frac{(b_1 + b_2 + \cdots + b_n)}{n}\\
&\geqslant \frac{a_1b_n + a_2b_{n-1} + \cdots + a_nb_1}{n}.
\end{aligned}$$

□

Exercises – Chebyshev Inequality.

33. (2002 TT† Northern Autumn SO Q4) If $x, y, z \in \mathbb{R}$ such that $0 < x, y, z < \pi/2$, prove

$$\frac{x\cos x + y\cos y + z\cos z}{x + y + z} \leqslant \frac{\cos x + \cos y + \cos z}{3}.$$

11.12 There's more than one way!

We prove

$$a^2 + b^2 + c^2 \geqslant ab + bc + ca, \quad \text{for all } a, b, c \in \mathbb{R},$$

four different ways, in order to demonstrate the usage of some inequalities.

Proof 1 (using $x^2 \geqslant 0, x \in \mathbb{R}$). Assume $a, b, c \in \mathbb{R}$. Then

$$(a - b)^2 \geqslant 0 \tag{11.12.1}$$
$$(b - c)^2 \geqslant 0 \tag{11.12.2}$$
$$(c - a)^2 \geqslant 0 \tag{11.12.3}$$

$$\begin{aligned}
\therefore\ & a^2 - 2ab + b^2\\
&+ b^2 - 2bc + c^2\\
&+ c^2 - 2ca + a^2 \geqslant 0, \qquad \text{adding (11.12.1)–(11.12.3)}\\
\therefore\ & 2a^2 + 2b^2 + 2c^2 \geqslant 2ab + 2bc + 2ca\\
\therefore\ & a^2 + b^2 + c^2 \geqslant ab + bc + ca
\end{aligned}$$

□

†TT (Tournament of the Towns).

Proof 2 (using AM-GM). Assume $a, b, c \in \mathbb{R}$. Then $a^2, b^2, c^2 \geqslant 0$, so that by AM-GM we have the following three inequalities:

$$\frac{a^2+b^2}{2} \geqslant \sqrt{a^2b^2} \tag{11.12.4}$$

$$\frac{b^2+c^2}{2} \geqslant \sqrt{b^2c^2} \tag{11.12.5}$$

$$\frac{c^2+a^2}{2} \geqslant \sqrt{c^2a^2} \tag{11.12.6}$$

$$\begin{aligned} \therefore\ a^2+b^2+c^2 &\geqslant |ab|+|bc|+|ca|, \text{ adding (11.12.4)–(11.12.6),}\\ &\qquad\qquad \text{noting that } \sqrt{x^2} = |x| \text{ for any } x \in \mathbb{R}.\\ &\geqslant ab+bc+ca \end{aligned}$$

□

Proof 3 (using Cauchy-Schwarz). Assume $a, b, c \in \mathbb{R}$ and let

$$\underset{\sim}{\boldsymbol{x}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad \underset{\sim}{\boldsymbol{y}} = \begin{pmatrix} b \\ c \\ a \end{pmatrix}$$

then

$$\begin{aligned} \left\| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\|^2 \cdot \left\| \begin{pmatrix} b \\ c \\ a \end{pmatrix} \right\|^2 &\geqslant \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} b \\ c \\ a \end{pmatrix} \right)^2 \\ (a^2+b^2+c^2)^2 &\geqslant (ab+bc+ca)^2 \\ a^2+b^2+c^2 &\geqslant |ab+bc+ca| \\ &\geqslant ab+bc+ca \end{aligned}$$

□

Proof 4 (using Rearrangement). W.l.o.g. assume $a \leqslant b \leqslant c$ then (vacuously)

$$a \leqslant b \leqslant c$$

so that with b, c, a as a permutation of a, b, c, using the latter part of the Rearrangement Inequality, we have

$$\begin{aligned} ab+bc+ca &\leqslant a^2+b^2+c^2 \\ \text{i.e. } a^2+b^2+c^2 &\geqslant ab+bc+ca. \end{aligned}$$

□

Exercises – Miscellaneous examples.

34. Prove that for any natural number $n \geqslant 2$,

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 1.$$

Hint: First use the observation that

$$\frac{1}{k} - \frac{1}{k+1} = \frac{k+1-k}{k(k+1)} = \frac{1}{k(k+1)}$$

to prove

$$\frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{(n-1)n} = \frac{n-1}{n}.$$

35. Prove that for any positive a and b

$$\frac{2}{\dfrac{1}{a} + \dfrac{1}{b}} \leqslant \sqrt{ab} \leqslant \frac{a+b}{2} \leqslant \sqrt{\frac{a^2+b^2}{2}}.$$

Problem Set 11.

1. For all $x_1, x_2, \ldots, x_n \in \mathbb{R}$ such that $x_i \geqslant i^2, i = 1, 2, \ldots, n$, prove that
$$\frac{x_1 + x_2 + \cdots + x_n}{2} \geqslant \sqrt{x_1 - 1^2} + 2\sqrt{x_2 - 2^2} + \cdots + n\sqrt{x_n - n^2}.$$

2. Prove that for all $x, y, z \in \mathbb{R}$,
$$x^2 + y^2 + z^2 - xy - yz - zx \geqslant \tfrac{3}{4}(x - y)^2.$$

3. For any positive $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in \mathbb{R}$, prove that
$$\sum_{i=1}^{n} \frac{1}{x_i y_i} \geqslant \frac{4n^2}{\sum_{i=1}^{n} (x_i + y_i)^2}.$$

4. If $0 < a, b, c \in \mathbb{R}$, show that
$$(ab)^2 + (bc)^2 + (ca)^2 \geqslant abc(a + b + c).$$

5. For all $x \in \mathbb{R}$, prove that
$$\frac{x^2 + 2}{\sqrt{x^2 + 1}} \geqslant 2.$$

6. For $a, b, c, d \geqslant 0$, prove that $\sqrt{(a + c)(b + d)} \geqslant \sqrt{ab} + \sqrt{cd}$.

7. Let $a, b, c > 0$. Show that
$$\frac{a + b + c}{abc} \leqslant \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

8. Show that, if $a, b > 0$ and $a + b = 1$ then
$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geqslant \frac{25}{2}.$$

9. For $x, y, z > 0$, prove that

   (a) $\dfrac{x^2}{y^2} + \dfrac{y^2}{z^2} + \dfrac{z^2}{x^2} \geqslant \dfrac{y}{x} + \dfrac{z}{y} + \dfrac{x}{z}$,

   (b) $\dfrac{x^2}{y^2} + \dfrac{y^2}{z^2} + \dfrac{z^2}{x^2} \geqslant \dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}$.

10. Prove that, if $a, b, c \in \mathbb{R}$ then
$$a^4(1 + b^4) + b^4(1 + c^4) + c^4(1 + a^4) \geqslant 6a^2b^2c^2,$$
and determine when equality occurs.

11. Let $1 < n \in \mathbb{N}$. Prove that
$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n^2} > 1.$$

12. Prove that, if $0 < a_1, a_2, \ldots, a_n \in \mathbb{R}$ such that $a_1 a_2 \cdots a_n = 1$ then
$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geqslant 2^n.$$

13. Prove that if $0 < a, b, c, d, e \in \mathbb{R}$ then
$$\left(\frac{a}{b}\right)^4 + \left(\frac{b}{c}\right)^4 + \left(\frac{c}{d}\right)^4 + \left(\frac{d}{e}\right)^4 + \left(\frac{e}{a}\right)^4 \geqslant \frac{b}{a} + \frac{c}{b} + \frac{d}{c} + \frac{e}{d} + \frac{a}{e}.$$

14. For all $1 \leqslant x_1, x_2, \ldots, x_n \in \mathbb{R}$, prove that
$$\frac{(1 + x_1)(1 + x_2) \cdots (1 + x_n)}{1 + x_1 x_2 \cdots x_n} \leqslant 2^{n-1}.$$

15. (1997 Melb. Uni. Maths Comp. Senior Q4) If $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$, show that
$$\frac{x}{1+y} + \frac{y}{1+x} \leqslant 1.$$

16. Let a, b, c be the side-lengths of a triangle. Prove that
$$\frac{a}{b+c-a} + \frac{b}{c+a-b} + \frac{c}{a+b-c} \geqslant 3.$$

17. (2002 Mentor Set) Let a, b, c be the side-lengths of a triangle. Prove that
$$(a+b-c)(b+c-a) + (b+c-a)(c+a-b) + (c+a-b)(a+b-c) \leqslant \sqrt{abc}(\sqrt{a} + \sqrt{b} + \sqrt{c}).$$

18. (2002 Mentor Set) For positive $a_1, a_2, \ldots, a_n \in \mathbb{R}$, $n \geqslant 2$, show that
$$(a_1^3 + 1)(a_2^3 + 1) \cdots (a_n^3 + 1) \geqslant (a_1^2 a_2 + 1)(a_2^2 a_3 + 1) \cdots (a_n^2 a_1 + 1).$$

19. (1974 USA MO Q2/1995 Canadian MO Q2) For positive $x, y, z \in \mathbb{R}$, prove that
$$x^x y^y z^z \geqslant (xyz)^{(x+y+z)/3}.$$

20. (1998 Canadian MO Q3) Let $n \in \mathbb{N}$ such that $n \geqslant 2$. Show that
$$\frac{1}{n+1}\left(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\right) \geqslant \frac{1}{n}\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right).$$

21. (1995 Irish MO Q6) Show that for all $n \in \mathbb{N}$,
$$n^n \leqslant (n!)^2 \leqslant \left(\frac{(n+1)(n+2)}{6}\right)^n.$$

22. (1997 Irish MO Q4) Let a, b, c be nonnegative real numbers such that $a + b + c \geqslant abc$. Prove that
$$a^2 + b^2 + c^2 \geqslant abc.$$

23. (2000 Irish MO Q6) Let $0 \leqslant x, y \in \mathbb{R}$ such that $x + y = 2$. Prove that

$$x^2y^2(x^2 + y^2) \leqslant 2.$$

24. (1990 British MO) For any positive $x, y, z \in \mathbb{R}$, prove that

$$\sqrt{x^2 - xy + y^2} + \sqrt{y^2 - yz + z^2} \geqslant \sqrt{z^2 + zx + x^2}.$$

25. (1997/8 Iranian MO Round 2 Q5) If $1 < x, y, z \in \mathbb{R}$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$, prove

$$\sqrt{x + y + z} \geqslant \sqrt{x - 1} + \sqrt{y - 1} + \sqrt{z - 1}.$$

26. For $0 < x \in \mathbb{R}$ and $n \in \mathbb{N}$, prove that

$$\frac{x^n}{1 + x + x^2 + \cdots + x^{2n}} \leqslant \frac{1}{2n + 1}.$$

27. (1976 Vietnam MO B3 Q6) For positive $x_1, x_2, \ldots, x_n \in \mathbb{R}$ such that $x_1 + x_2 + \cdots + x_n = 1$ and nonnegative $k \in \mathbb{Z}$, prove that

$$\frac{1}{x_1^k} + \frac{1}{x_2^k} + \cdots + \frac{1}{x_n^k} \geqslant n^{k+1}.$$

28. (1975 Swedish MO Q3) For positive $a, b, c \in \mathbb{R}$ and $n \in \mathbb{N}$, prove that

$$a^n + b^n + c^n \geqslant ab^{n-1} + bc^{n-1} + ca^{n-1}.$$

29. Prove that for all $a \geqslant b \geqslant 0$,

$$\frac{(a - b)^2}{8a} \leqslant \frac{a + b}{2} - \sqrt{ab} \leqslant \frac{(a - b)^2}{8b}.$$

30. Let $0 < x, y, z \in \mathbb{R}$. Prove that

$$\frac{x}{x + \sqrt{(x + y)(x + z)}} + \frac{y}{y + \sqrt{(y + z)(y + x)}} + \frac{z}{z + \sqrt{(z + x)(z + y)}} \leqslant 1.$$

31. (1989 USSR MO Q21) Find the least value of $(x + y)(y + z)$, given that $0 < x, y, z \in \mathbb{R}$ such that $xyz(x + y + z) = 1$.

32. (1991 USSR MO Q9) For all nonnegative a, b, c, prove that

$$\frac{(a + b + c)^2}{3} \geqslant a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}.$$

33. (1992 CIS‡ MO Q1) Prove that for all positive $a, b, c \in \mathbb{R}$,

$$a^4 + b^4 + c^2 \geqslant 2\sqrt{2}abc.$$

‡CIS (Commonwealth of Independent States) MO, was previously the USSR MO.

34. (1992 CIS[§] MO Q9) Prove that for any $a > 1$, $b > 1$,

$$\frac{a^2}{b-1} + \frac{b^2}{a-1} \geqslant 8.$$

35. (2002 TT Northern Autumn SO Q4) If $x, y, z \in \mathbb{R}$ such that $0 < x, y, z < \pi/2$, prove

$$\frac{x\cos x + y\cos y + z\cos z}{x+y+z} \leqslant \frac{\cos x + \cos y + \cos z}{3}.$$

36. (1986 AMO) Given $1 < n \in \mathbb{N}$ and $0 < a \in \mathbb{R}$, determine the maximum value of

$$\sum_{i=1}^{n-1} x_i x_{i+1}$$

taken over all sets of n nonnegative numbers x_i with sum a.

37. (1987 AMO) Prove that for each $n \in \mathbb{N}$ such that $n > 1$,

$$\sqrt{n+1} + \sqrt{n} - \sqrt{2} > 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}.$$

38. (1992 AMO) Let $n \in \mathbb{N}$. Show that

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} > n(\sqrt[n]{2} - 1).$$

39. (1992 AMO) Let $n \in \mathbb{N}$, $0 < a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $s = a_1 + a_2 + \cdots + a_n$. Prove that

$$\sum_{i=1}^{n} \frac{a_i}{s - a_i} \geqslant \frac{n}{n-1} \text{ and } \sum_{i=1}^{n} \frac{s - a_i}{a_i} \geqslant n(n-1).$$

40. (1997 AMO Q2) Let $a_1, a_2, \ldots, a_k \in \mathbb{R}$ satisfying

    (i) $0 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$, and

    (ii) $a_1 + a_2 + \cdots + a_k = 1$.

    Prove that $\dfrac{a_1 + a_2 + \cdots + a_n}{n} \leqslant \dfrac{1}{k}$ for $n = 1, 2, \ldots, k$.

41. (1997 AMO Q7) Let $1 < m, n \in \mathbb{Z}$. Prove that

$$\frac{1}{\sqrt[m]{n+1}} + \frac{1}{\sqrt[n]{m+1}} > 1.$$

42. (1998 AMO Q6) Prove that for any $n \in \mathbb{N}$,

$$(1998n)! \leqslant \left(\frac{3995n+1}{2} \cdot \frac{3993n+1}{2} \cdot \frac{3991n+1}{2} \cdot \cdots \cdot \frac{n+1}{2}\right)^n.$$

[§]CIS (Commonwealth of Independent States) MO, was previously the USSR MO.

43. (1999 AMO Q5) Let $1 < x \in \mathbb{R}$ and $1 < n \in \mathbb{N}$. Prove that
$$1 + \frac{x-1}{nx} < \sqrt[n]{x} < 1 + \frac{x-1}{n}.$$

44. (2001 AMO Q4) Prove the polynomial $4x^8 - 2x^7 + x^6 - 3x^4 + x^2 - x + 1$ has no real root.

45. (1990 APMO Q2) Let $0 < a_1, a_2, \ldots, a_n \in \mathbb{R}$ and let S_k be the sum of all products of $a_1, a_2, \ldots, a_n$ taken k at a time. Show that
$$S_k S_{n-k} \geqslant \binom{n}{k}^2 a_1 a_2 \cdots a_n,$$
for $k = 1, 2, \ldots, n-1$.

46. (1991 APMO Q3) Let $0 < a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in \mathbb{R}$ such that
$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k.$$
Show that
$$\sum_{k=1}^{n} \frac{(a_k)^2}{a_k + b_k} \geqslant \tfrac{1}{2} \sum_{k=1}^{n} a_k.$$

47. (1996 APMO) Let $m, n \in \mathbb{N}$ such that $n \leqslant m$. Prove that
$$2^n n! \leqslant \frac{(m+n)!}{(m-n)!} \leqslant (m^2 + m)^n.$$

48. (1996 APMO) Let a, b, c be the side-lengths of a triangle. Prove that
$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leqslant \sqrt{a} + \sqrt{b} + \sqrt{c},$$
and determine when equality occurs.

49. (1998 APMO Q3) Let $0 < a, b, c \in \mathbb{R}$. Prove that
$$\Bigl(1 + \frac{a}{b}\Bigr)\Bigl(1 + \frac{b}{c}\Bigr)\Bigl(1 + \frac{c}{a}\Bigr) \geqslant 2\Bigl(1 + \frac{a+b+c}{\sqrt[3]{abc}}\Bigr).$$

50. (2002 APMO) Positive real numbers x, y, z satisfy $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$. Prove that
$$\sqrt{x+yz} + \sqrt{y+zx} + \sqrt{z+xy} \geqslant \sqrt{xyz} + \sqrt{x} + \sqrt{y} + \sqrt{z}.$$

51. (1990 IMO Short List Q23) If $0 \leqslant w, x, y, z \in \mathbb{R}$ such that $wx + xy + yz + zw = 1$, prove
$$\frac{w^3}{x+y+z} + \frac{x^3}{y+z+w} + \frac{y^3}{z+w+x} + \frac{z^3}{w+x+y} \geqslant \frac{1}{3}.$$

52. (1993 IMO Short List Q24) For positive $a, b, c, d \in \mathbb{R}$, show that
$$\frac{a}{b+2c+3d}+\frac{b}{c+2d+3a}+\frac{c}{d+2a+3b}+\frac{d}{a+2b+3c} \geqslant \frac{2}{3}.$$

53. (1998 IMO Short List A1) If $0 < x_1, x_2, \ldots, x_n \in \mathbb{R}$ such that $x_1+x_2+\cdots+x_n < 1$, prove
$$\frac{x_1x_2\cdots x_n(1-x_1-x_2-\cdots-x_n)}{(x_1+x_2+\cdots+x_n)(1-x_1)(1-x_2)\cdots(1-x_n)} \leqslant \frac{1}{n^{n+1}}.$$

54. (1964 IMO Q2) Let a, b, c be the side-lengths of a triangle. Prove that
$$a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c) \leqslant 3abc.$$

55. (1975 IMO Q1) Let $x_i, y_i \in \mathbb{R}$ for $i = 1, 2, \ldots, n$, such that $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$ and $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n$. Prove that, if $z_1, z_2, \ldots, z_n$ is any permutation of $y_1, y_2, \ldots, y_n$, then
$$\sum_{i=1}^{n}(x_i-y_i)^2 \leqslant \sum_{i=1}^{n}(x_i-z_i)^2.$$

56. (Adapted from 1978 IMO Q5) Let $a_1, a_2, \ldots, a_n$ be a sequence of distinct positive integers. Prove that for all $n \in \mathbb{N}$,
$$\sum_{k=1}^{n}\frac{a_k}{k^2} \geqslant \sum_{k=1}^{n}\frac{1}{k}.$$

57. (1983 IMO Q6) Let a, b, c be the side-lengths of a triangle. Prove that
$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a) \geqslant 0.$$
When does equality hold?

58. (1995 IMO Q2) Let $0 < a, b, c \in \mathbb{R}$ such that $abc = 1$. Prove that
$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)} \geqslant \frac{3}{2}.$$

59. (2000 IMO) Let $0 < a, b, c \in \mathbb{R}$ such that $abc = 1$. Prove that
$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \leqslant 1.$$

60. (2001 IMO Q2) For all positive $a, b, c \in \mathbb{R}$, prove that
$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}} \geqslant 1.$$