CHAPTER 15

Invariants

Sometimes a problem can be easily solved once one has an **invariant**. Generally, an *invariant* is a *property* that is satisfied by a class of mathematical objects that remains unchanged when transformations of a certain type are applied to the objects.

In a problem that involves a number of steps, if one can show a certain property is satisfied at every step, often by induction, then one can often deduce a solution to a problem by checking satisfaction of the property at the last step.

The idea is best demonstrated with some examples.

1. (2006 Pólya) The first 1998 positive integers are listed on the blackboard. Two randomly selected numbers are erased and their difference is written on the board. The step is repeated until only one number is left on the board.

   Is the remaining number odd or even?

   Solution. Observe that we start with 999 odd numbers and 999 even numbers. Let us consider what cases can occur at each step.

   Case 1: Two even numbers are replaced. The replacement number is also even. So the number of even numbers has reduced by 1, and the number of odd numbers remains the same.

   Case 2: Two odd numbers are replaced. The replacement number is even. So the number of even numbers has increased by 1, and the number of odd numbers has reduced by 2.

   Case 3: One odd number and one even number are replaced. The replacement number is odd. So the number of even numbers has decreased by 1, and the number of odd numbers is the same as it was previously.

   Observe that in all cases, the number of odd numbers remains odd after each step, and of course the total number of numbers reduces by 1 at each step. So an *invariant* for each step of this problem is:

   The number of odd numbers is odd.

   Thus, after 1997 steps there will be one number left, and since after each step the number of odd numbers is odd, that last number remaining is odd.

2. (2009 AMO Q3) The polynomials $x^2 + x$ and $x^2 + 2$ are written on a white board. Sue is allowed to write on this board the sum, the difference or the product of any two polynomials already on the board. She repeats this process as many times as she likes.

   Can Sue ever write the polynomial x on the board?

   Solution. Define a polynomial $p(x)$ to have Property P if

   $$6 \mid p(2).$$

   Observe that for

   $$f(x) = x^2 + x \text{ and } g(x) = x^2 + 2,$$

   we have

   $$6 \mid f(2) \text{ and } 6 \mid g(2),$$

   since $f(2) = 6 = g(2)$. So each of the initial polynomials has Property P.

Lemma. *If f, g are on the board and $6 \mid f(2)$ and $6 \mid g(2)$ then $6 \mid (f+g)(2)$, $6 \mid (f-g)(2)$, and $6 \mid (f \cdot g)(2)$.*

Proof. This follows immediately from the definitions of $f+g$, $f-g$ and $f \cdot g$:

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(f \cdot g)(x) = f(x) \cdot g(x)$$

So, for example, since $6 \mid f(2)$ and $6 \mid g(2)$, we have $6 \mid \big(f(2) + g(2)\big) = (f+g)(2)$. □

Put another way, the Lemma says:

> If two chosen polynomials on the board have Property P then the resultant polynomial has Property P.

Thus, it follows, by induction, that at every stage, every polynomial on the board has Property P, i.e. P is an *invariant*.
Suppose $h(x) = x$ is written on the board at some stage. Then it must have Property P. But $h(2) = 2$ and $6 \nmid 2$. So $h(x)$ does not have Property P (a contradiction).
Thus Sue can never write x on the board.

3. (2009 AMO trial) In a latest theory of particle physics there are three fundamental particles. When two particles of different types collide they are replaced by a particle of the third type. Two particles of the same type never collide. Prove that if an experiment begins with an equal number of particles of each type it cannot end with just one particle remaining.

   Solution. We are done if we can find a property that is invariant over each step of the experiment. Observe that the numbers of two types of particles decreases by 1 and the other type increases by 1, so that the total number of particles decreases by 1 at each step. So the experiment proceeds for only a finite number of steps, terminating when there is only one type of particle remaining.

   Since at each step, the numbers of each type of particles changes by 1, the parity (i.e. what it is modulo 2) of the numbers of each type of particle changes at each step. If (ℓ, m, n) represents that there are ℓ, m, n of the three types of particle, respectively, then the initial state is (N, N, N) for some $N \in \mathbb{N}$. Thus initially, the numbers (and hence the parities of those numbers) of each particle type are the same, and since all the particle types swap parity at each step, we have an invariant property, namely that: the numbers of particles of each type have the same parity.

   At each step, the state parity toggles between (even, even, even) and (odd, odd, odd). Since $(0, 0, 1)$ (and any permutation) is of neither parity type, the experiment can never finish with just one particle.

Exercise Set 15.

1. (2012 TT Northern Autumn JO Q3) A 10×10 table is filled out according to the rules of the 'Minesweeper' game: each cell either contains a mine or a number that shows how many mines are in neighbouring cells, where cells are neighbours if they have a common edge or vertex.

   If all mines are removed from the table and then new mines are placed in all previously mine-free cells, with the remaining cells to be filled out with the numbers according to the 'Minesweeper' game rule as above, can the sum of all numbers in the table increase?