CHAPTER 9

The Pigeon-Hole Principle

The *Pigeon-Hole Principle (PHP)* is easily illustrated by a simple example:

> If 5 pigeons fly into 4 pigeon-holes then at least one pigeon-hole contains two or more pigeons.

i.e. if there is at least one more *pigeon* than there are *pigeon-holes* then at least one of the *pigeon-holes* has more than one *pigeon*.

Example. *Suppose that in my dresser drawer I have socks of three colours ... loose. A bit silly, because I have to get up this morning while it's still dark. How do I ensure that I get a matching pair of socks in the most economical way ... without disturbing my partner?*

Solution. I take 4 socks from the drawer ... since then, by PHP, I must have at least one pair. The idea is that the *colours* (3 of them) are the *pigeon-holes* and the *socks* are the *pigeons*. □

Of course, this idea can be generalised a bit:

Theorem (Extended Pigeon-Hole Principle). *If there are N* pigeon-holes *and more than kN* pigeons *then at least one* pigeon-hole *has at least $k+1$* pigeons*.*

Proof. Assume there are N pigeon-holes and $\geqslant kN+1$ *pigeons*, and for a contradiction, suppose each pigeon-hole has $\leqslant k$ pigeons. Then

$$\begin{aligned}\#\text{pigeons} &\leqslant kN\\ &< kN+1 \ \lightning\end{aligned}$$

So, in fact, there is a pigeon-hole with (at least) $k+1$ pigeons. □

Elections! Elections!

Voting for a member of the House of Representatives

Members for the House of Representatives are elected using the *preferential voting system*. Suppose that in a given *electorate* there are six *candidates*. Then, a *formal vote* (i.e. a vote that obeys the rules and so one that will be counted) numbers the candidates from 1 to 6 in some order.

So ... how is the winning candidate determined? Well ... first the number 1 votes are counted. In our example, this results in 6 piles of ballot papers: one for each candidate. The candidate with the least number 1 votes is then excluded; and that candidate's pile of ballot papers are re-distributed to the other 5 piles according to the number 2 votes on those ballot papers.

Question 9.1. How many of the number 1 (*primary*) votes must a candidate get to ensure they are not excluded at the first round?

At the second round the candidate with the smallest pile of ballot papers after the first round is excluded; and that candidate's pile of ballot papers are re-distributed to the remaining (4, in our example) piles according to the *next* number preference on those ballot papers, i.e. the *smallest* number vote that doesn't correspond to an excluded candidate – which will be either number 2 votes or number 3 votes.

Subsequent rounds are analogous to the second round. The natural conclusion of this process is a single pile corresponding to the winning candidate.

Question 9.2. How many of the number 1 (*primary*) votes must a candidate get to ensure they are not excluded at the kth round, (where k is less than the number of candidates)?

Question 9.3. At the k^{th} round, how big must a candidate's pile be at the beginning of the round to ensure they are not excluded at that round, (where k is less than the number of candidates)?

In answering these questions you will see a number of short-cuts to the process described above, e.g. if after any round a candidate's pile contains more than half the number of ballot papers then that candidate is certainly the winner.

Question 9.4. Can you think of another short-cut ... using the PHP?

Voting for a member of the Senate

Usually at a *general election* there is a *half-senate* election, i.e. as well as voting for all the *House of Representatives* we vote for *half* the Senate, the other half keep their jobs until there is another general election.* On 2 March 1996, in Western Australia there were 29 candidates for the 6 (of the 12) senator positions that had become vacant. So a *formal* Senate vote numbered the candidates in some order from 1 to 29.†

So ... how are the winning 6 candidates determined? Well ... first as for the *House of Representatives* the number 1 votes are counted. In the recent election, this would have resulted in 29 piles of formal ballot papers. Let V be the total number of *formal votes.* Then a candidate is elected once their pile of ballot papers achieves a *quota*,

$$Q = \left\lfloor \frac{V}{6+1} \right\rfloor + 1$$

of the formal votes cast.

Question 9.5. What is the significance of the *quota* Q? (Ideas explored with regard to House of Representatives voting should help you.)

Now ... what happens? Like the *House of Representatives* there are a number of *rounds*, but unlike the *House of Representatives* each round has two parts. Firstly, any candidate who has achieved the quota Q is declared elected. Then the ballot papers of these elected candidates are re-distributed to the next not-so-far elected preference but at a *reduced* value: they are scaled by the factor‡

$$\frac{c - Q}{c},$$

where c is the number of ballot papers in the candidate's pile. Once no more candidates can be lifted to a quota this way, the second part of the round begins. This proceeds *exactly* in

*Occasionally, exceptional circumstances bring about a *double dissolution*, where both houses of parliament are dissolved and there is a *full-senate* election.

†Voters had to either put a 1 in one box *above the line* or to number all boxes below the line. Each party corresponding to a box above the line logged with the Electoral Commission how *they* would number the boxes *below the line*. So really all *formal* votes cast, number the 29 candidates.

‡The idea is that the *surplus* votes for elected candidates are passed on to the remaining candidates; but it would be *unfair* to simply take any $c - Q$ votes as the surplus. So *all* c votes are re-distributed but at a reduced weighting.

the way a *House of Representatives* round does: the candidate with the smallest pile of ballot papers is excluded and those ballot papers are re-distributed to the remaining piles according to the *next* preference on those ballot papers, at *full* value (where "*next*" preference this time means the least number vote that corresponds neither to an *already-elected* candidate nor to an *excluded* candidate).

Now, to be convinced that this is a viable means of voting we really need only consider the following two questions:

Question 9.6. Why can no more than 6 be elected in this way?

Question 9.7. Why are (at least) 6 candidates elected by this method ... unless there is a tie?

Exercise Set 9.

1. In a group of 8 people show that at least two have their birthday on the same day of the week.

2. Three natural numbers are chosen at random. Their sum is 19. Show at least one number is 7 or more.

3. A box contains 10 French books, 20 Spanish books, 8 German books, 15 Russian books, and 25 Italian books. How many must we choose to ensure that we have 12 books in the same language?

4. There are 30 students in a class. While doing a keyboarding test one student made 14 mistakes, while the rest made fewer mistakes. Show that at least 3 students made the same number of mistakes.

5. A teacher starts each year with 3 jokes. Over 12 years the teacher never repeated the same triple of jokes. What is the smallest number of jokes the teacher must have in her repertoire for this to be possible?

6. A canteen has 95 tables with a total of 465 chairs. Can we be sure that there is a table with at least 6 chairs?

7. Prove that of any 5 points chosen in an equilateral triangle of side-length 1, there are two points whose distance apart is at most $\frac{1}{2}$.

8. Suppose we have 27 *distinct* positive odd numbers ... all less than 100. Show there is a pair of numbers whose sum is 102.

9. The integers 1 to 10 are arranged in random order around a circle. Show that there are three consecutive numbers whose sum is *at least* 17.

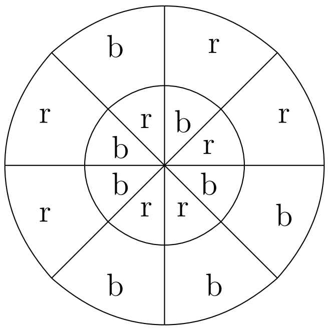
10. Six swimmers training together either swam in a race or watched the others swim. At least how many races must have been scheduled if every swimmer had opportunity to watch all of the others?

11. There are 11 people at a party. Some of them exchange handshakes with some of the others. Prove that at least two people have shaken the same number of hands.

12. A computer is used for 99 billable hours over a period of 12 days. Prove that on some pair of consecutive days the computer was used at least 17 billable hours.
    Note. If the actual usage is x hours, the *billable hours* are $\lceil x \rceil$.

13. Show that given any 17 natural numbers it is possible to choose 5 whose sum is divisible by 5.

14. A circle is divided into 8 equal sectors. Half are coloured red and half are coloured blue. A smaller circle is also divided into equal sectors, half coloured red and half coloured blue. The smaller circle is placed concentrically on the larger. Prove that no matter how the red and blue sectors are chosen it is always possible to rotate the smaller circle so that at least 4 colour matches are obtained. (The diagram below shows an example.)



15. Five microcomputers are to be connected to three printers. How many connections are necessary between computers and printers in order to ensure that whenever any three computers require a printer the printers are available?

16. Prove that, of any 5 points chosen within a square of side-length 2, there are two whose distance apart is at most $\sqrt{2}$.

17. A disk of radius 1 is completely covered by 7 identical smaller disks. (They may overlap.) Show that the radius of each of the smaller disks must not be less than $\frac{1}{2}$.

18. A *graph* consists of *vertices* (singular: *vertex*) and *edges*. *Vertices* are usually represented by filled-in dots and each *edge* starts and finishes at a *vertex*. The *degree* of a *vertex* is the number of *edges* that start (or finish) at that *vertex*.
    Suppose a *graph* has 9 *vertices* such that each vertex has *degree* 5 or 6. Prove that at least 5 vertices have degree 6 or at least 6 vertices have degree 5.

19. How many trees can farmer Fred plant on his 100 m square field if they are to be no closer than 10 m apart? (Neglect the thickness of the trees.)