CHAPTER 12

Plane Geometry

12.1 Introduction

We adopt common conventions with regard to notation. **Points** are denoted by capitals $A, B, C, \dots$. Given two points A, B there is exactly one **line** joining them denoted by AB. A **line** is a *straight line* that is infinite in extent in both directions. In other contexts, AB may denote the **line segment** of points between A and B, or the **ray** (or half-line) that starts at A and passes through B. The length of the line segment AB is usually also denoted by AB, but sometimes by $|AB|$. Given two rays AB and AC starting from the common point A, the angle they form is denoted by $\angle BAC$ or $\angle CAB$, or by just $\angle A$ if there is only one (interior) angle formed at the point A. Sometimes lines are denoted by lowercase letters $\ell, m, n, \dots$. Lengths are sometimes denoted by lowercase Roman letters, e.g. $a, b, c, \dots$, especially in triangles where a represents the length of the side opposite angle A, b represents the length of the side opposite angle B, etc. Sizes of angles are commonly represented by lowercase Greek letters $\alpha, \beta, \dots$.

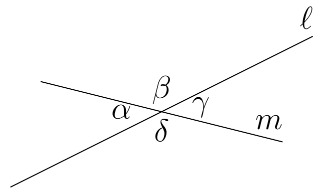
Ideally the theorems should be ordered so that the later ones follow from those proved earlier. Our ordering is close to such an ordering, but at times we compromise when a natural grouping of theorems dictates a different ordering.

12.2 Lines and angles

We take as an axiom that the angle at a point on a straight line is a constant regardless of the point or line. Such an angle is called a **straight angle** and its measure is 180°.

When two lines intersect, four angles are formed. Two such angles are called **vertically opposite** (or just **opposite**) if they are not formed on the same side of one of the lines. The straight angle axiom (postulate) implies the following theorem.

Theorem 12.2.1. *The opposite angles formed by intersecting straight lines are equal. In the diagram, $\alpha = \gamma$ and $\beta = \delta$.*



Proof. Let ℓ, m be intersecting straight lines, and let α, γ be opposite angles, with β an angle adjacent to both α and γ. Then $\alpha + \beta$ and $\gamma + \beta$ are straight angles. By the straight angle postulate,

$$\alpha + \beta = \gamma + \beta$$
$$\alpha = \gamma, \qquad \text{adding } -\beta \text{ to both sides.}$$ □

12.3 Congruence of triangles

Definition 12.3.1. Two polygons are said to be **congruent**, if their corresponding sides and corresponding angles are equal.

When we say two *triangles* ABC and XYZ are *congruent* we mean that the correspondence of vertex A to X, B to Y and C to Z determines the congruence.

We denote that two triangles ABC and XYZ are *congruent* by writing $\triangle ABC \cong \triangle XYZ$.

Triangles may be determined to be congruent by rules known by the initialisms: SAS, SSS, ASA, and RHS. More precisely, these rules are as per the following theorem.

Theorem 12.3.2 (SAS, SSS, ASA, RHS Rules). *If, for two triangles,*

SAS: *two sides and the included angle of one triangle are equal to the two sides and the included angle of the other,*

or

SSS: *three sides of one triangle are equal to the three sides of the other,*

or

ASA: *two angles and the included side of one triangle are equal to the two angles and the included side of the other,*

or

RHS: *the hypotenuse and one other side of a right-angled triangle are equal to the hypotenuse and one side of the other right-angled triangle,*

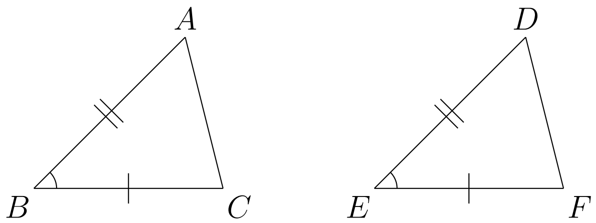
then the triangles are congruent.

Definition 12.3.3. A triangle is **isosceles** if two of its sides are equal. By convention, the common vertex of the two equal sides of an *isosceles* triangle is written between the other two vertices, i.e. to say $\triangle XYZ$ is isosceles we imply that $YX = YZ$.

Theorem 12.3.4. *If a triangle is* isosceles *then the angles opposite the equal sides are equal. Conversely, if two angles of a triangle are equal then the two sides opposite the equal angles are equal, so that the triangle is isosceles.*

To prove the two theorems of this section, it's convenient to do so in this order: *SAS Rule*, Theorem 12.3.4, *SSS Rule*, *ASA Rule*. For now we omit the proof of the *RHS Rule*.

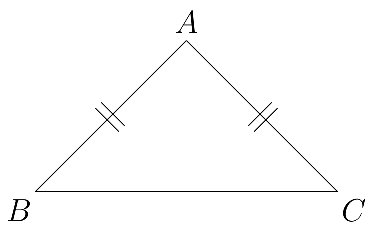
Proof of Theorem 12.3.2 SAS Rule.
In the triangles ABC and DEF we have side $AB = DE$, included angle $\angle ABC = \angle DEF$, and side $BC = EF$. We must show $\triangle ABC \cong \triangle DEF$.



Place triangle ABC over triangle DEF so that B falls on E and edge BC runs along line EF. Since $BC = EF$, C falls on F. Since $\angle ABC = \angle DEF$, line BA falls on ED, and since $AB = DE$, A fall on D. Since A falls on D and C falls on F, line segment AC falls on DF. Hence $\triangle ABC \cong \triangle DEF$. □

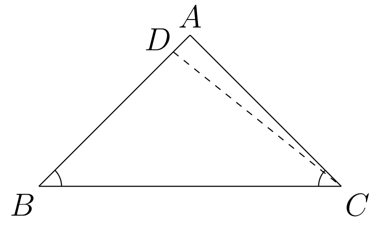
Proof of Theorem 12.3.4.
($\Longrightarrow$) Assume in triangle ABC that $AB = AC$. Then

$AB = AC,$	given	(12.3.1)
$\angle BAC = \angle CAB,$	same angle	
$AC = AB,$	equivalent to (12.3.1)	
$\triangle ABC \cong \triangle ACB,$	by SAS Rule	
$\therefore \angle ABC = \angle ACB$		



So we have shown that an isosceles triangle has the angles opposite its equal sides equal.

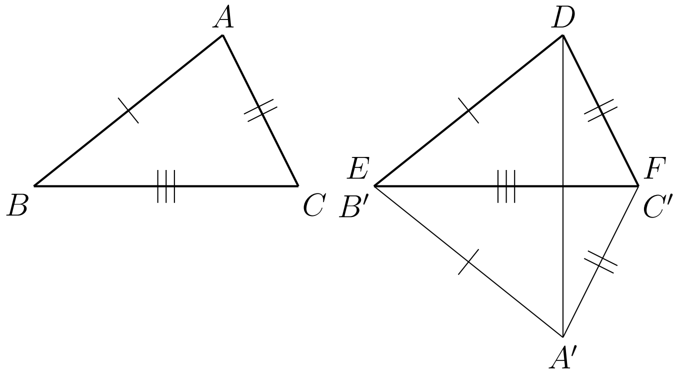
(⟸) Now assume in $\triangle ABC$ that $\angle ABC = \angle ACB$. Along the ray BA, construct (by compass) the point D such that $DB = AC$. Now we have



$$\begin{aligned} DB &= AC, && \text{by construction} \\ \angle ABC = \angle DBC &= \angle ACB, && \text{given} \\ BC &= CB, && \text{same line segment} \\ \triangle DBC &\cong \triangle ACB, && \text{by SAS Rule} \\ \therefore \angle DCB &= \angle ABC = \angle ACB \end{aligned}$$

Thus line DC coincides with line AC. Hence $D = A$, and $AB = DB = AC$. So we have shown that a triangle with two angles equal has the sides opposite the equal angles equal. □

Proof of Theorem 12.3.2 SSS Rule. Assume in triangles ABC and DEF that $AB = DE$, $BC = EF$ and $CA = FD$. Transport triangle ABC so that B falls on E and line BC runs along EF. Since $BC = EF$, C falls on F. Now let triangle ABC fall on the opposite side of line EF to triangle DEF so that A falls on A'. The transported copy of $\triangle ABC$ is $\triangle A'B'C'$ in the diagram.



By construction, $\triangle ABC \cong \triangle A'B'C'$, where $E = B'$ and $F = C'$. In particular, $A'E = AB = DE$ and $A'F = AC = DF$, so that triangles DEA' and DFA' are isosceles. So by Theorem 12.3.4, we have

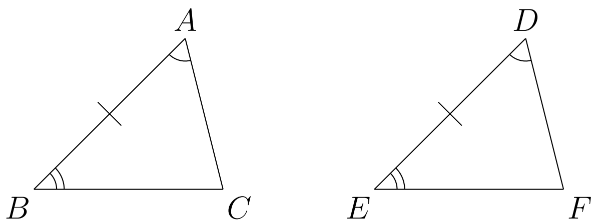
$$\angle EDA' = \angle EA'D \quad \text{and} \quad \angle FDA' = \angle FA'D.$$

Hence,

$$\angle EDF = \angle EDA' + \angle FDA' = \angle EA'D + \angle FA'D = \angle EA'F = \angle BAC.$$

So now $\triangle ABC \cong \triangle DEF$ by the SAS Rule. □

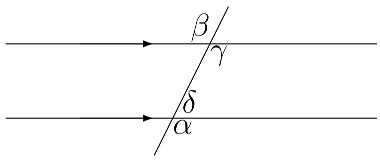
Proof of Theorem 12.3.2 ASA Rule. In the triangles ABC and DEF we have $\angle ABC = \angle DEF$, included side $AB = DE$ and $\angle BAC = \angle EDF$. Place $\triangle ABC$ over $\triangle DEF$ so that A falls on D and AB runs along DE. Since $AB = DE$, B falls on E. Also AC runs along DF because $\angle BAC = \angle EDF$. Similarly, BC runs along EF because $\angle ABC = \angle DEF$.



Thus the intersection point C of AC and BC must fall on the intersection point F of EF and DF. So $\triangle ABC$ is exactly superimposed over $\triangle DEF$, and hence $\triangle ABC \cong \triangle DEF$. □

12.4 Parallel lines

In the diagram the two horizontal lines are **parallel**. The line cutting the parallel lines is called a **transversal**. Angles α and β are called **alternate angles**, α and γ are **corresponding angles**, and angles α and δ are **supplementary angles**. Alternate angles and corresponding angles are equal, and pairs of supplementary angles sum to 180°.



12.5 Similarity of triangles

Definition 12.5.1. Two polygons are said to be **similar** (denoted by $\sim$), if

(i) corresponding sides are in the same proportion, and

(ii) corresponding angles are equal.

As with congruence, when we say two triangles ABC and XYZ are *similar* we mean that the correspondence of vertex A to X, B to Y and C to Z determines the similarity. We denote that two triangles ABC and XYZ are *similar* by writing $\triangle ABC \sim \triangle XYZ$.

Triangles may be determined to be similar by rules known by the initialisms: PAP, PPP, AA, and PPA. Each rule corresponds to a congruence rule, with side-length proportionality replacing equality. More precisely, these rules are as per the following theorem.

Theorem 12.5.2 (PAP, PPP, AA, PPA Rules). *If, for two triangles,*

PAP: *two sides of one of the triangles are in the same proportion to the two sides of the other, and the included angles between each pair of sides are equal,*
or

PPP: *three sides of one of the triangles are in the same proportion to the three sides of the other,*
or

AA: *two angles of one of the triangles equal two angles of the other,*
or

PPA: *two sides of one of the triangles are in the same proportion to the two sides of the other, and a corresponding non-included, non-acute angle of each triangle are equal,*

then the triangles are similar.

Theorem 12.5.3. *If a line joins the midpoints of two sides of a triangle then that line is parallel to the third side and its length is equal to one half of the length of the third side.*

Theorem 12.5.4. *A line parallel to one side of a triangle divides the other two sides in the same proportion.*

Theorem 12.5.5. *The bisector of one side of a triangle divides the opposite side in the same ratio as the other two sides.*

12.6 More triangle theorems

Theorem 12.6.1. *The sum of the interior angles of a triangle is* 180°.

Theorem 12.6.2. *An exterior angle of a triangle equals the sum of the two non-adjacent interior angles.*

12.7 Angles of a convex polygon

Theorem 12.7.1. *The sum of the interior angles of an n-sided convex polygon is* $180(n-2)^\circ$.

Proof. Let an interior point of the polygon be O. Construct line segments from the vertices of the polygon to O. The polygon is now divided into n triangles. The angle sum of the polygon is thus equal to the angle sum of the triangles *minus* the total of the angles around O, namely

$$180n^\circ - 360^\circ = 180(n-2)^\circ. \qquad \square$$

12.8 Quadrilaterals

Theorem 12.8.1. *The opposite sides and opposite interior angles of a parallelogram are equal.*

Theorem 12.8.2. *If a quadrilateral has opposite sides equal then it is a parallelogram.*

Theorem 12.8.3. *If a quadrilateral has opposite interior angles equal then it is a parallelogram.*

Theorem 12.8.4. *The diagonals of a parallelogram bisect each other.*

Theorem 12.8.5. *The diagonals of a rhombus are perpendicular.*

Also see the Theorem 12.12.10 on *cyclic quadrilaterals* in the *Circles* section.

12.9 Special Triangle Theorems

Theorem 12.9.1 (Pythagoras' Theorem). *In a right triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides.*

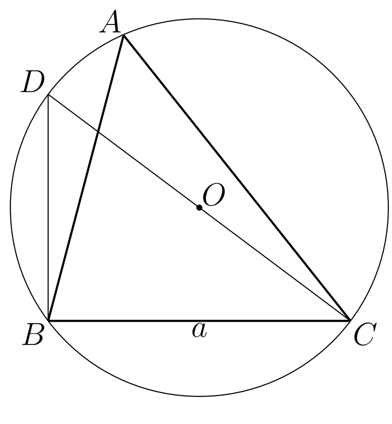
Note that the next theorem is 'out of order'; it depends on Theorems 12.12.2 and 12.12.10.

Theorem 12.9.2 (Sine Rule). *In a triangle ABC where a, b, c are the lengths of the sides opposite the vertices A, B, C, respectively,*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

where R is the circumradius of $\triangle ABC$.

Proof. Around the $\triangle ABC$ we draw its circumcircle, with circumcentre at O (initially assumed to be inside $\triangle ABC$), and radius R. Produce CO to meet the circumference at D, so that CD is a diameter; and then draw chord DB. Now $\angle CBD = 90^\circ$ since it is inscribed in a semicircle. So



$$\sin D = \frac{a}{CD} = \frac{a}{2R}.$$

But $\angle D = \angle A$, since both are inscribed in the arc BC. Thus we have $\sin A = \sin D = a/(2R)$, or equivalently

$$\frac{a}{\sin A} = 2R.$$

By symmetry, we also have

$$\frac{b}{\sin B} = 2R \quad \text{and} \quad \frac{c}{\sin C} = 2R.$$

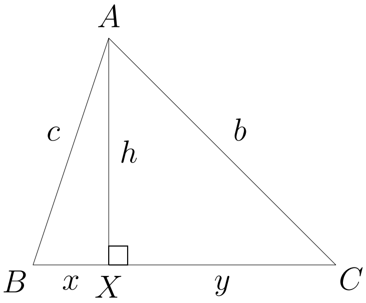
So we have proved the result for the case where O is inside $\triangle ABC$.

If O is outside $\triangle ABC$, producing CO to D as before, $CDBA$ is a cyclic quadrilateral, so that $\angle A$ and $\angle D$ are supplementary, whence $\sin A = \sin D$ and hence the result still follows. $\square$

Theorem 12.9.3 (Cosine Rule). *In a triangle ABC where a, b, c are the lengths of the sides opposite the vertices A, B, C, respectively,*

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Proof. With $\triangle ABC$ as per the diagram, with altitude $h = AX$, $x = BX$, $y = XC$, we have

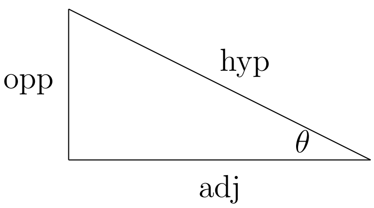


$$\begin{aligned} c^2 &= x^2 + h^2 \\ &= (a-y)^2 + h^2 \\ &= a^2 + y^2 + h^2 - 2ay \\ &= a^2 + b^2 - 2ab\cos C. \end{aligned}$$

□

12.10 Essential Trigonometry

Standard functions and their reciprocals:



$$\boxed{\sin\theta = \frac{\text{opp}}{\text{hyp}}} \qquad \boxed{\operatorname{cosec}\theta = \frac{1}{\sin\theta}}$$

$$\boxed{\cos\theta = \frac{\text{adj}}{\text{hyp}}} \qquad \boxed{\sec\theta = \frac{1}{\cos\theta}}$$

$$\boxed{\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sin\theta}{\cos\theta}} \qquad \boxed{\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}}$$

Observe that each (function, reciprocal function) pair has one function whose name starts with `co` and one that doesn't.

Trig. versions of Pythagoras' Theorem:

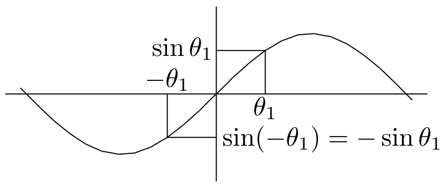
$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

$$\boxed{\tan^2\theta + 1 = \sec^2\theta}$$

$$\boxed{1 + \cot^2\theta = \operatorname{cosec}^2\theta}$$

The second equation is obtained from the first by dividing throughout by $\cos^2\theta$.
The third equation is obtained from first by dividing throughout by $\sin^2\theta$.

Oddness and evenness:



$$\boxed{\sin(-\theta) = -\sin\theta} \qquad \boxed{\sin \text{ is } \textit{odd}}$$

$$\boxed{\cos(-\theta) = \cos\theta} \qquad \boxed{\cos \text{ is } \textit{even}}$$

$$\boxed{\tan(-\theta) = -\tan\theta} \qquad \boxed{\tan \text{ is } \textit{odd}}$$

Complementary angles:

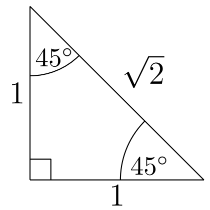
$$\boxed{\sin(90^\circ - \theta) = \cos\theta}$$

$$\boxed{\cos(90^\circ - \theta) = \sin\theta}$$

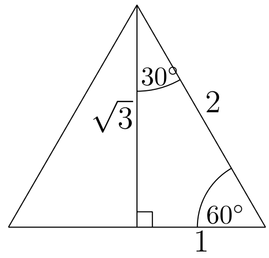
$$\boxed{\tan(90^\circ - \theta) = \cot\theta = \frac{1}{\tan\theta}}$$

Standard values:

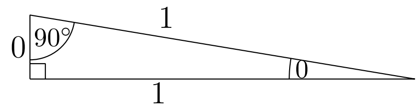
Isosceles right-angled triangle



Half equilateral triangle



Degenerate triangle



θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	$0 = \sqrt{\frac{0}{4}}$	$\frac{1}{2} = \sqrt{\frac{1}{4}}$	$\frac{1}{\sqrt{2}} = \sqrt{\frac{2}{4}}$	$\frac{\sqrt{3}}{2} = \sqrt{\frac{3}{4}}$	$1 = \sqrt{\frac{4}{4}}$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Sine Rule:

Theorem. *For $\triangle ABC$, with sides opposite angles $\angle A, \angle B, \angle C$ equal to a, b, c, respectively, and circumradius R,* $\boxed{\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R}$.

Periodicity: The *period* of sin and cos is 360°, and of tan it is 180°, i.e.

$$\boxed{\sin(\theta + 360°) = \sin\theta}$$

$$\boxed{\cos(\theta + 360°) = \cos\theta}$$

$$\boxed{\tan(\theta + 180°) = \tan\theta}$$

Symmetries: By sketching the graphs of $\sin\theta, \cos\theta, \tan\theta$ many symmetries are apparent, e.g.

$\boxed{\sin\theta = \sin(180° - \theta)}$ (symmetry about $\theta = 90°$)

$\boxed{\cos\theta = \cos(360° - \theta)}$ (symmetry about $\theta = 180°$)

Angle sum and difference identities:

$$\boxed{\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta}$$

$$\boxed{\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta}$$

$$\boxed{\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

$$\boxed{\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta}$$

$$\boxed{\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}}$$

$$\boxed{\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}}$$

Double angle identities: These follow from the angle sum identities by putting $\beta = \alpha$.

$$\boxed{\sin(2\alpha) = 2\sin\alpha\cos\alpha}$$

$$\boxed{\begin{aligned}\cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\ &= 1 - 2\sin^2\alpha \\ &= 2\cos^2\alpha - 1\end{aligned}}$$

$$\boxed{\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha}}$$

12.11 Areas and perimeters

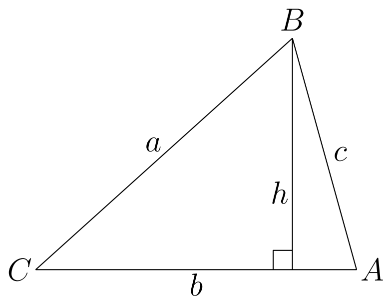
Notation 12.11.1. If a closed figure is denoted by $XY\dots Z$ then its area is denoted by $|XY\dots Z|$, i.e. by enclosing in vertical lines.

Theorem 12.11.2. *The area of a parallelogram is equal to bh where b is the length of its base and h is its height (the perpendicular distance from the base to the parallel side opposite).*

Theorem 12.11.3. *(a) The area of a triangle is equal to $\frac{1}{2}bh$ where b is the length of its base and h is its height (the perpendicular distance from the base to the vertex opposite).*

(b) Let the triangle be ABC, labelled in the standard way, and with vertex B opposite the base (which is thus labelled b). Then the area of $\triangle ABC$,

$$|ABC| = \tfrac{1}{2}ab\sin C.$$

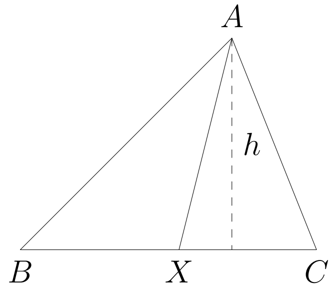


Proof of (b). Let the sides and height of $\triangle ABC$ be as labelled in the diagram. Then

$$\begin{aligned}\frac{h}{a} &= \sin C \\ \therefore h &= a\sin C \\ |ABC| &= \tfrac{1}{2}bh \\ &= \tfrac{1}{2}ba\sin C.\end{aligned}$$ □

Definition 12.11.4. In a triangle any side can be considered a **base**, and we refer to the vertex opposite such a side, as an **apex relative to** that chosen **base**. Similarly, an *altitude* dropped from an apex to its corresponding base, is an **altitude relative to** that **base**, e.g. in $\triangle ABC$, A is the *apex relative to base BC*, B is the *apex relative to base CA*, and the altitude emanating from A is the *altitude relative to base BC*.

Theorem 12.11.5. *If triangles $\triangle_1, \triangle_2$ (with areas $|\triangle_1|, |\triangle_2|$) have bases b_1, b_2 respectively along a common line and share an apex relative to bases b_1, b_2, then $|\triangle_1| : |\triangle_2| = b_1 : b_2$.*



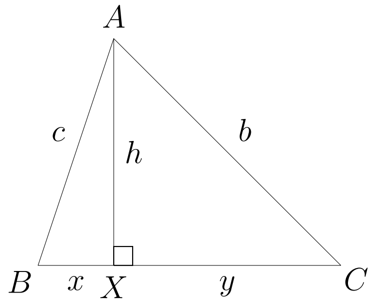
Proof. Let $\triangle_1 = \triangle ABX$, $\triangle_2 = \triangle AXC$.
Then $\triangle_1, \triangle_2$ have common apex A relative to bases $b_1 = BX$, $b_2 = XC$ along a common line

$$\begin{aligned}&\implies \triangle_1, \triangle_2 \text{ have common altitude } h \text{ dropped from } A \text{ to } BC \\ &\implies |\triangle_1| : |\triangle_2| = |ABX| : |AXC| \\ &\qquad\qquad\qquad = \tfrac{1}{2}b_1h : \tfrac{1}{2}b_2h \\ &\qquad\qquad\qquad = b_1 : b_2\end{aligned}$$ □

Theorem 12.11.6 (Heron's Theorem). *For $\triangle ABC$ with sides a, b, c and* semiperimeter $s = (a+b+c)/2$, *its area*

$$|ABC| = \sqrt{s(s-a)(s-b)(s-c)}.$$

Proof. With $\triangle ABC$ as per the diagram, with altitude $h = AX$, $x = BX$, $y = XC$, we have



$$\begin{aligned}
c^2 - x^2 &= h^2 \\
&= b^2 - (a-x)^2 \\
c^2 - b^2 &= x^2 - (a^2 - 2ax + x^2) \\
&= -a^2 + 2ax \\
2ax &= a^2 - b^2 + c^2 \\
x &= \frac{a^2 - b^2 + c^2}{2a} \\
h^2 &= c^2 - \frac{(a^2 - b^2 + c^2)^2}{4a^2} \\
|ABC|^2 &= \tfrac{1}{4}a^2h^2 \\
&= \tfrac{1}{4}a^2\left(c^2 - \frac{(a^2 - b^2 + c^2)^2}{4a^2}\right) \\
&= \tfrac{1}{16}\left(4a^2c^2 - (a^2 - b^2 + c^2)^2\right) \\
&= \tfrac{1}{16}\left((2ac)^2 - (a^2 - b^2 + c^2)^2\right) \\
&= \tfrac{1}{16}\left(2ac + (a^2 - b^2 + c^2)\right)\left(2ac - (a^2 - b^2 + c^2)\right) \\
&= \tfrac{1}{16}(a^2 + 2ac + c^2 - b^2)\left(b^2 - (a^2 - 2ac + c^2)\right) \\
&= \tfrac{1}{16}\left((a+c)^2 - b^2\right)\left(b^2 - (a-c)^2\right) \\
&= \tfrac{1}{16}(a+c+b)(a+c-b)(b+a-c)(b-a+c) \\
&= \tfrac{1}{16} \cdot 2s \cdot (2s-2b)(2s-2c)(2s-2a), \text{ where } 2s = a+b+c \\
&= s(s-b)(s-c)(s-a) \\
|ABC| &= \sqrt{s(s-b)(s-c)(s-a)}.
\end{aligned}$$

□

Theorem 12.11.7. *The area of a circle of radius r is πr^2 and its circumference is $2\pi r$.*

12.12 Circles

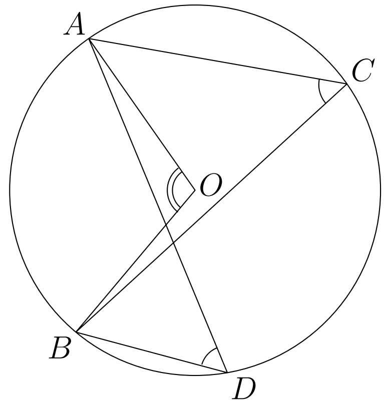
Theorem 12.12.1. *There is a unique circle through any triple of non-collinear points.*

Proof. Let the three non-collinear points be A, B, C. The points are distinct since two distinct points are sufficient to define a line, so that if any points are coincident then the points would be collinear, contradicting their non-collinearity. Form the perpendicular bisector of each of AB and BC. (Recall that the perpendicular bisector of two points is the locus of points that are equidistant from two given points.) These bisectors are non-parallel, since A, B, C are non-collinear. Hence the bisectors intersect. Let the point of intersection be O. Then $OA = OB$ since O lies on the bisector of AB. Similarly, $OB = OC$ since O lies on the bisector of BC. Thus

$$OA = OB = OC$$

so that O is equidistant from A, B and C. Hence we A, B and C lie on a circle with centre O and radius OA. Since O is the unique intersection of the perpendicular bisectors of AB and BC the circle through A, B and C is unique. □

Theorem 12.12.2. *If AB is an arc of a circle then angles subtended at the circumference opposite AB are equal and are equal to half the angle subtended at the centre, i.e. in the diagram $\angle ACB = \angle ADB = \frac{1}{2}\angle AOB$.*



Proof. Construct OC, forming isosceles triangles AOC and BOC. Let the equal base angles of $\triangle AOC$ be x and the equal base angles of $\triangle BOC$ be y. Then

$$\begin{aligned}\angle ACB &= x + y\\ \angle AOB &= 360^\circ - (180^\circ - 2x) - (180^\circ - 2y)\\ &= 2x + 2y = 2\angle ACB\\ \therefore \angle ACB &= \tfrac{1}{2}\angle AOB\end{aligned}$$

Suppose that C is in fact at D, and x and y are defined as before. Then $\angle ADB = y - x$ and

$$\begin{aligned}\angle AOB &= 180^\circ - 2x - (180^\circ - 2y)\\ &= 2(y - x) = 2\angle ACB\end{aligned}$$

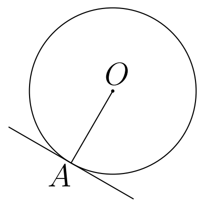
with the same conclusion as before. □

Theorem 12.12.3. *If AB is a semicircular arc of a circle and C is any point on the circumference of the circle then $\angle ACB$ is a right angle.*

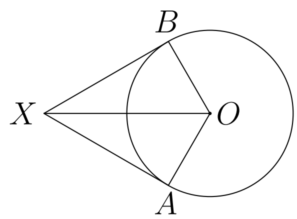
Theorem 12.12.4. *If A and B are points on the circumference of a circle with centre O and C is an exterior point of the circle such that BC is a tangent to the circle then $\angle ABC = \frac{1}{2}\angle AOB$.*

Theorem 12.12.5. *A line from the centre of a circle perpendicular to a chord bisects the chord and its arc.*

Theorem 12.12.6. *A line meeting a circle at a point A is tangent to the circle if and only if the radius to the point of contact with the line at A is perpendicular to the line.*



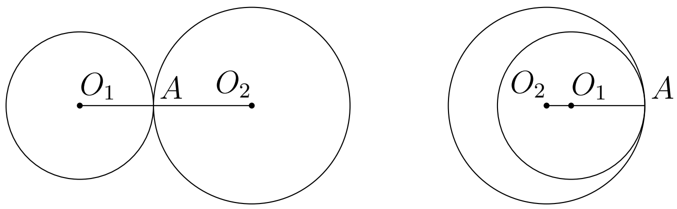
Theorem 12.12.7. *The two tangents drawn to a circle from an exterior point of the circle have the same length. In the diagram, $XA = XB$.*
Moreover, the line joining the centre of the circle and the exterior point bisects the angle between the two tangents. In the diagram, OX bisects $\angle AXB$.



Proof.

$OA = OB$,	(radii of same circle)
$90^\circ = \angle OAX = \angle OBX$,	by Theorem 12.12.6
OX is common	
$\therefore \triangle OAX \cong \triangle OBX$,	by the RHS Rule
$\therefore XA = XB$	
and $\angle OXA = \angle OXB$,	i.e. OX bisects $\angle AXB$. □

Theorem 12.12.8. *If two circles touch at a single point then this point and the centres of the circles are collinear. Below, O_1, O_2 and the point of contact A of the circles are collinear.*

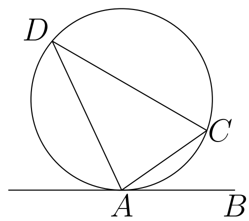


Theorem 12.12.9. *If two circles intersect at two points then the line through their centres. is the perpendicular bisector of their common chord.*

Theorem 12.12.10. *Opposite angles of a cyclic quadrilateral sum to* $180°$ *and if a pair of opposite angles of a quadrilateral sum to* $180°$ *then it is cyclic.*

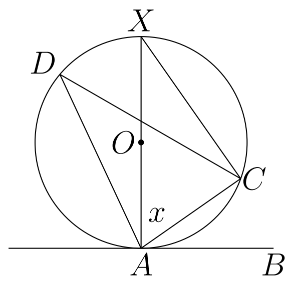
Theorem 12.12.11. *The centre of the circumcircle of a triangle is the intersection of the perpendicular bisectors of the sides of the triangle.*

Theorem 12.12.12 (Tangent-chord Theorem or Alternate Segment Theorem). *Let AC be a chord in a circle and let AB be a line meeting the circle at A. Then AB is tangent to the circle if and only if $\angle CAB = \angle ADB$ for any point D on the arc AC of the circle opposite B.*



Proof. Draw a diameter from A through the centre O (to X) and let $x = \angle XAC$. Then

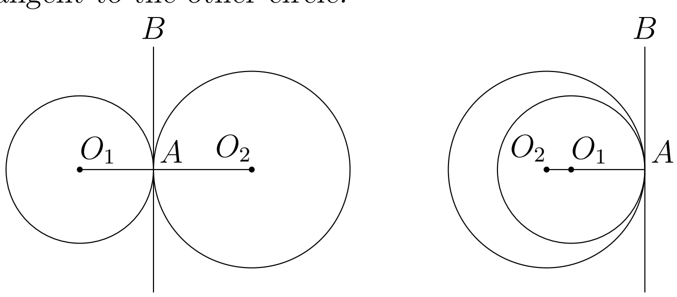
$$\begin{aligned}
\angle XAB &= 90°, && \text{by Theorem 12.12.6, } OA \perp AB \\
\therefore \angle CAB &= \angle XAB - \angle XAC \\
&= 90° - x \\
\angle ACX &= 90°, && \text{by Theorem 12.12.3, since } AX \text{ is a diameter} \\
\therefore \angle AXC &= 180° - \angle ACX - \angle XAC \\
&= 90° - x \\
&= \angle CAB \\
\angle ADC &= \angle AXC, && \text{by Theorem 12.12.2, common arc: } AC \\
\therefore \angle ADC &= \angle CAB.
\end{aligned}$$



□

Remark. In the diagram of the proof above, if we think of X moving around the circumference of the circle, we always have $\angle AXC = \angle ADC$, by Theorem 12.12.2. As X moves around toward A, $\angle BAC$ can be thought of as the *limit* of $\angle AXC$ as $X \to A$, since the chord XA (extended) becomes the tangent AB when X and A become the one point.

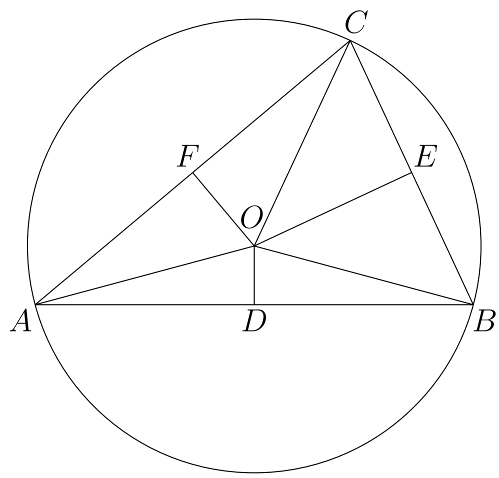
Theorem 12.12.13. *If two circles are tangential at a point A then the tangent to one of the circles at A is also a tangent to the other circle.*



Theorem 12.12.14. *For any triangle ABC, the perpendicular bisectors of the three sides AB, BC and CA are concurrent, at some point O. Furthermore, $AO = BO = CO$, so that O is the centre of a circle K of radius $R = AO$ that passes through each of the vertices A, B, C of $\triangle ABC$.*
(K, O, R are respectively the ***circumcircle***, ***circumcentre*** *and* ***circumradius*** *of $\triangle ABC$, and K is* ***circumscribed*** *about $\triangle ABC$.)*

Proof. Draw $\triangle ABC$ and let the midpoints of sides AB, BC and CA be D, E and F, respectively. Draw the perpendicular bisectors from sides AB and BC to meet at a point O, and join O to F (we must show OF is also a perpendicular bisector). Then

$$\begin{aligned}
AD &= BD \\
90^\circ = \angle ODA &= \angle ODB \\
OD &\text{ is common} \\
\therefore \triangle ODA &\cong \triangle ODB, && \text{by the SAS Rule} \\
\therefore AO &= BO \\
\text{Similarly, } \triangle OEB &\cong \triangle OEC \\
\therefore BO &= CO.
\end{aligned}$$

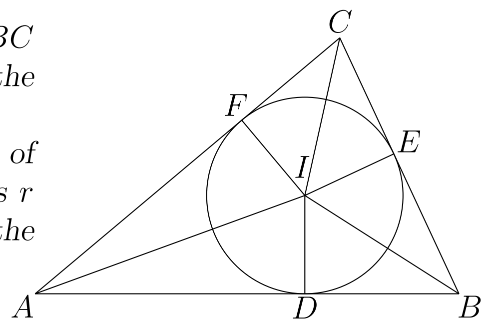


So now we have

$$\begin{aligned}
AO &= CO \\
AF &= CF \\
OF &\text{ is common} \\
\therefore \triangle OFA &\cong \triangle OFB, && \text{by the SSS Rule} \\
\therefore \angle OFA &= \angle OFC = 90^\circ, && \text{since } \angle AFC \text{ is a straight angle } (180^\circ)
\end{aligned}$$

Thus, OF is the perpendicular bisector of AC, and hence the perpendicular bisectors of $\triangle ABC$ concur at O. Also, we showed that $AO = BO = CO$, so that A, B and C lie on a circle centred at O with radius $R = AO$. □

Theorem 12.12.15. *The angle bisectors of any triangle ABC are concurrent, at some point I that is equidistant from the sides of $\triangle ABC$.*
(Let r be the common distance of I from the three sides of $\triangle ABC$. Then the circle $K(I, r)$ with centre I and radius r is ***inscribed*** *in $\triangle ABC$, and $K(I, r), I, r$ are respectively the* ***incircle***, ***incentre*** *and* ***inradius*** *of $\triangle ABC$.)*



12.13 Glossary

Acute An angle is **acute** if it is smaller than a right angle.

Altitude The line through a vertex of a triangle that is perpendicular to the opposite side. A triangle has three **altitude**s; they are concurrent, meeting at the triangle's **orthocentre**.

Arc Any portion of the circumference of a circle.

Centroid The point at which the three medians of a triangle concur. The **centroid** trisects each of the medians, i.e. splits each median in the ratio $2:1$. More generally, the *centroid* of a figure is its **centre of mass**.

Cevian A line segment in a triangle joining a vertex and a point on the side opposite the vertex.

Chord A line segment whose endpoints lie on the circumference of a circle.

Circumcentre, circumcircle The three perpendicular bisectors of the sides of a triangle concur at the **circumcentre** of the triangle, which is the centre of the **circumcircle**, the circle that passes through the three vertices of the triangle.

Collinear This means *lying on the same straight line.* Several points are **collinear** if you can draw a single straight line through all of them.

Complementary angles A pair of angles whose sum is $90°$.

Concurrent This means *going through the same point.* Several lines are **concurrent** if they all intersect in the same point.

Congruent Two polygons are **congruent** if they have the same size and shape (i.e. if one were to shift and/or reflect one polygon the vertices of the two polygons could be made to line up exactly); in particular corresponding sides are of the same length.

Convex A set S of points on a line, plane or in space is **convex** if for any points A, B in S, all points on the line segment AB are in S. We say a polygon is *convex* if any line segment between points on the boundary of the polygon only intersects the interior of the polygon, i.e. all its interior angles are less than $180°$, e.g. any regular polygon is convex.

Cyclic A quadrilateral is **cyclic** if a circle may be drawn that passes through each of its four vertices.

Diameter A chord of a circle that passes through the circle's centre.

Edge A side of a geometrical figure, or more generally, a line segment that joins two vertices.

Equilateral A triangle is **equilateral** if all its sides are of equal length. An equilateral triangle necessarily has all its angle equal to $60°$.

Euler line The line in a triangle on which the *orthocentre, centroid* and *circumcentre* lie.

Hypotenuse The side opposite the right angle of a right triangle.

Incentre, incircle, inradius The three internal bisectors of the angles of a triangle concur at the **incentre** of the triangle, which is the centre of the **incircle**, the circle that touches each side of the triangle, i.e. each side of the triangle is a tangent to the incircle. The radius of the *incircle* is the triangle's **inradius**.

Isosceles A triangle is **isosceles** if two of its sides are of equal length, in which case, the two angles not included by the sides of equal length are equal.

Line In plane geometry, a **line** always means a *straight line* that is infinite in both directions.

Line segment A piece of a line of a definite length with two ends.

Locus (plural: **loci**) The line, curve or region traced out by a point satisfying certain conditions, e.g. if a point moves with fixed distance from a fixed point then its **locus** is a circle.

Median A line joining the vertex of a triangle to the midpoint of the opposite side. A triangle has three **median**s; they concur at the *centroid* of the triangle.

Medial triangle of a triangle ABC. Triangle formed by joining the midpoints of the sides of $\triangle ABC$.

Nine-point circle The feet of the three altitudes of a triangle ABC (i.e. the vertices of its *orthic triangle*), the midpoints of the sides of $\triangle ABC$ (i.e. the vertices of its *medial triangle*), and the midpoints of the line segments from the vertices of $\triangle ABC$ to the *orthocentre* of $\triangle ABC$, lie on the same circle; this circle is known as the *nine-point circle* of $\triangle ABC$. Its radius is $\frac{1}{2}R$, where R is the radius of the *circumcircle* of $\triangle ABC$. Its centre is the midpoint of the *Euler line* of $\triangle ABC$.

Obtuse An angle is *obtuse* if is larger than a right angle and smaller than a straight angle.

Orthogonal Same as *perpendicular*.

Orthocentre The common intersection point of the three altitudes of a triangle.

Orthic triangle of a triangle ABC. Triangle formed by joining the feet of the altitudes of $\triangle ABC$.

Parallel Two lines are **parallel** if they never meet.

Parallelogram A quadrilateral that has two pairs of parallel sides.

Pedal point, pedal triangle A **pedal point** is a point P inside a triangle ABC from which perpendiculars are dropped to the three sides of $\triangle ABC$. A triangle formed by joining the feet of the three perpendiculars dropped from a pedal point is called a **pedal triangle**. The **orthic triangle** is the *pedal triangle* formed when the *pedal point* P is the *orthocentre* of $\triangle ABC$. The **medial triangle** is the *pedal triangle* formed when the *pedal point* P is the *circumcentre* of $\triangle ABC$. In the case where P lies on the *circumcircle* of $\triangle ABC$, the feet Q, R, S of the perpendiculars to the sides of $\triangle ABC$ are collinear, so that the 'pedal triangle' formed is degenerate; the line through Q, R, S, in this case is known as a **simson**.

Perpendicular At right angles.

Plane figure A geometrical figure consisting of vertices and edges that can be drawn in the plane; a 2-dimensional object.

Polygon A plane figure whose edges are connected end to end in a loop. A **polygon** with n sides is sometimes called an n-**gon**. (Technically, a *gon* is an angle, but an n-*gon* has just as many sides as it has angles, so could just as easily have been called an n-lateral.) *Trigon* and *trilateral* are uncommon synonyms for *triangle*. 4-gons are generally referred to as *quadrilaterals* and sometimes as *quadrangles*. And we have *pentagon* (5-gon), *hexagon* (6-gon), *heptagon* (7-gon), *octagon* (8-gon), *nonagon* (9-gon), *decagon* (10-gon), *dodecagon* (12-gon), etc.

Point of Contact The common point of a tangent with a circle, or of two circles that are tangential.

Quadrangle, quadrilateral A polygon with 4 sides (and therefore 4 angles).

Radius (plural: **radii**) A line segment from the centre to the circumference of a circle.

Ray The part of a line that lies on one side of a point.

Reflex angle An angle that is larger than a straight angle but less than a full rotation, i.e. an angle of size between 180° and 360°.

Regular A polygon is **regular** if all its sides are equal and all its angles are equal.

Rhombus A parallelogram whose sides are all of equal length.

Right angle Half a straight angle. Its measure is 90°.

Right triangle, right-angled triangle A triangle with a right angle.

Secant A line that intersects a circle in two distinct points. A **chord** is just the segment of a secant that joins the two points of intersection with the circle.

Sector The area bounded by an arc of a circle and the two radii joining the arc.

Similar Two polygons are **similar** if angles at corresponding vertices are equal (if the two polygons are $ABC\ldots$ and $XYZ\ldots$ then A corresponds to X, B corresponds to Y, etc.), in which case corresponding sides are in the same proportion.

Simple A **simple** plane figure is one that does not cross itself.

Simson line, simson If P lies on the *circumcircle* of a triangle ABC then the feet Q, R, S of the perpendiculars drawn to the (extensions of the) sides of $\triangle ABC$ are *collinear*. The line through Q, R and S is the **Simson line** or **simson** of the point P with respect to triangle $\triangle ABC$. Also see *pedal point*.

Straight angle The angle at a point on (either side of) a straight line. Its measure is 180°.

Supplementary angles A pair of angles whose sum is 180°.

Tangent A line in the same plane as a circle that intersects (i.e. touches) the circle at exactly one point.

Tangential, touch A line and a circle, or two circles, are **tangential** (or **touch**) if they intersect at exactly one point.

Transversal A line that intersects two or more parallel lines.

Trapezium, trapezoid A quadrilateral that has one pair of opposite sides parallel.

Vertex (plural: **vertices**) A "corner" of a geometrical figure, i.e. a point at which edges meet.

12.14 Ceva's Theorem

Definition 12.14.1. A **cevian** of a triangle is a line segment joining a vertex of the triangle to a point on the opposite side. Thus, if X is a point on BC of $\triangle ABC$, then AX is a *cevian*. Medians, altitudes and angle bisectors are all examples of cevians.

Lemma 12.14.2 (Addendo). $k = \dfrac{a}{b} = \dfrac{c}{d}$ *where* $b, d, b+d \neq 0 \implies k = \dfrac{a+c}{b+d}$.

Proof. Assume $k = a/b = c/d$ and $b, d, b+d \neq 0$. Then

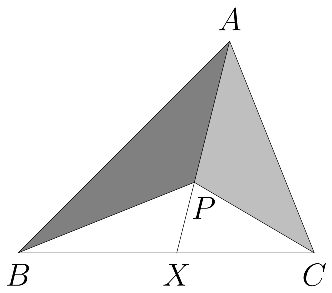
$$\begin{aligned} a &= kb \text{ and } c = kd \\ \implies \frac{a+c}{b+d} &= \frac{kb+kd}{b+d} \\ &= \frac{k(b+d)}{b+d} \\ &= k. \end{aligned}$$

□

Corollary 12.14.3. $k = \dfrac{a}{b} = \dfrac{c}{d}$ *where* $b, d, b-d \neq 0 \implies k = \dfrac{a-c}{b-d}$.

Proof. Observe that $c/d = (-c)/(-d)$. Then the result follows from Lemma 12.14.2 with c replaced by $-c$ and d replaced by $-d$. □

Lemma 12.14.4 (Ceva's Lemma). *If P is a point other than A on cevian AX of $\triangle ABC$ then $|ABP| : |CAP| = BX : XC$.*
In this case, we say $\triangle ABP$ and $\triangle CAP$ are above BX and XC.



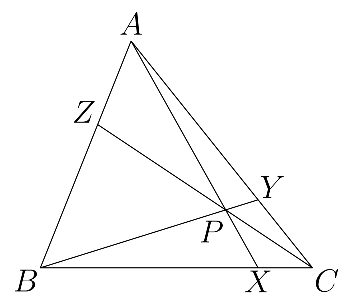
Proof. Since triangle pairs $\triangle ABX, \triangle AXC$ and $\triangle PBX, \triangle PXC$ have apices A and P, respectively, relative to bases BX and XC along a common line, by Theorem 12.11.5,

$$\begin{aligned} \frac{|ABX|}{|AXC|} &= \frac{BX}{XC}, \text{ and} \\ \frac{|PBX|}{|PXC|} &= \frac{BX}{XC} \\ \therefore\ \frac{|ABP|}{|CAP|} = \frac{|ABX| - |PBX|}{|AXC| - |PXC|} &= \frac{BX}{XC}, \text{ by Corollary 12.14.3, since } A \neq P \end{aligned}$$

i.e. $|ABP| : |CAP| = BX : XC$. □

Theorem 12.14.5 (Ceva's Theorem). *Let AX, BY, CZ be cevians of $\triangle ABC$. Then*

$$AX, BY, CZ \text{ concur} \iff \frac{BX}{XC}\,\frac{CY}{YA}\,\frac{AZ}{ZB} = 1.$$



Proof. ($\Longrightarrow$) Assume AX, BY, CZ of $\triangle ABC$ concur at a point P. Then in accordance with Lemma 12.14.4,

$\triangle ABP$ and $\triangle CAP$ are *above* BX and XC,
$\triangle BCP$ and $\triangle ABP$ are *above* CY and YA, and
$\triangle CAP$ and $\triangle BCP$ are *above* AZ and ZB.

Hence, by Lemma 12.14.4,

$$\frac{|ABP|}{|CAP|} = \frac{BX}{XC},$$
$$\frac{|BCP|}{|ABP|} = \frac{CY}{YA},$$
$$\frac{|CAP|}{|BCP|} = \frac{AZ}{ZB}.$$

So we have,

$$\begin{aligned}\frac{BX}{XC}\frac{CY}{YA}\frac{AZ}{ZB} &= \frac{|ABP|}{|CAP|}\cdot\frac{|BCP|}{|ABP|}\cdot\frac{|CAP|}{|BCP|}\\ &= 1.\end{aligned}$$

($\Longleftarrow$) Now suppose $\frac{BX}{XC}\frac{CY}{YA}\frac{AZ}{ZB} = 1$, and let P be the point of intersection of AX and BY. Let CP meet AB at Z'. Then by the forward argument we have

$$\frac{BX}{XC}\frac{CY}{YA}\frac{AZ'}{Z'B} = 1$$

and hence we have

$$\frac{AZ}{ZB} = \frac{AZ'}{Z'B}$$

so that both Z and Z' divide AB in the same ratio and must therefore be the same point. The result follows. □

12.15 The Euler Line

The **Euler line** of a triangle is the line segment joining its *circumcentre* and its *orthocentre*. What's particularly interesting is that the *centroid* also lies on this line:

Theorem 12.15.1. *Let the circumcentre, orthocentre and centroid of a triangle be O, H and G, respectively. Then O, H and G are collinear, and G divides the* Euler line *HO in the ratio* $2:1$, *i.e.* $HG : GO = 2 : 1$.

12.16 The Nine-point Circle

The Orthic Triangle. Let D, E, F be the feet of the altitudes from A, B, C, respectively, of $\triangle ABC$. Then $\triangle DEF$ is the **orthic triangle** of $\triangle ABC$.
The Medial Triangle. Let A', B', C' be the midpoints of sides BC, CA, AB, respectively, of $\triangle ABC$. Then $\triangle A'B'C'$ is the **medial triangle** of $\triangle ABC$.

Theorem 12.16.1 (Nine-point Circle). *The feet of the three altitudes of a triangle, the midpoints of the three sides, and the midpoints of the line segments from the vertices to the orthocentre, all lie on the circle with centre the centre of the Euler Line and with radius* $\frac{1}{2}R$*, where* R *is the circumradius, i.e. if in the notation above,* $\triangle ABC$ has orthic triangle $\triangle DEF$, medial triangle $\triangle A'B'C'$, H *is the orthocentre, and, further,* K, L, M *are the midpoints of line segments* AH, BH, CH, *respectively, and* N *is the midpoint of the Euler Line, then* $\triangle DEF$, $\triangle A'B'C'$ *and* $\triangle KLM$ *have the same circumcircle centred at* N *and of radius* $\frac{1}{2}R$*. Moreover,* $\triangle A'B'C' \cong \triangle KLM$ *and a half-turn (a rotation through* $180°$ *takes* $\triangle A'B'C'$ *onto* $\triangle KLM$*.*

12.17 The Radical Axis of Two Circles

Theorem 12.17.1 (Radical Axis of Two Circles). *The locus of points whose powers with respect to two non-concentric circles are equal is a line perpendicular to the line joining the centres of the circles. This line is called the* ***radical axis*** *of the two circles.*

12.18 Power of a Point

Definition 12.18.1. For a circle K of radius R and a point P of distance d from the centre of K, the **power** $\mathcal{P}(P, K)$ of P with respect to the circle K is

$$\mathcal{P}(P, K) = d^2 - R^2.$$

Theorem 12.18.2 (Power of a point). *If a line through a point* P *meets* K *a circle at points* A *and* A' *then the product*

$$PA \times PA'$$

is the ***power*** *of* P *with respect to the circle, i.e. if* P *is distance* d *from the centre of the circle* K*,*

$$PA \times PA' = d^2 - R^2.$$

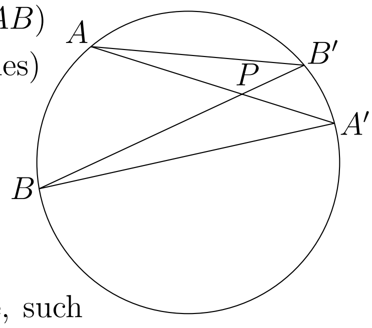
Directed segment convention. *By adopting the convention that line segments are* ***directed****, the above equation accounts for the* ***power*** *being positive (*PA *and* PA' *are in the same direction), zero, or negative (*PA *and* PA' *are in opposite directions), according to whether* P *is outside, on, or inside the circle, respectively.*

The above result follows from the next theorem.

Theorem 12.18.3 (Bowtie Theorem). *If two lines through a point* P *meet a circle at points* A, A' *and* B, B'*, respectively, then*

$$PA \times PA' = PB \times PB'.$$

Proof. We have two main cases to consider: P inside the circle and P outside the circle. First suppose P is inside the circle.

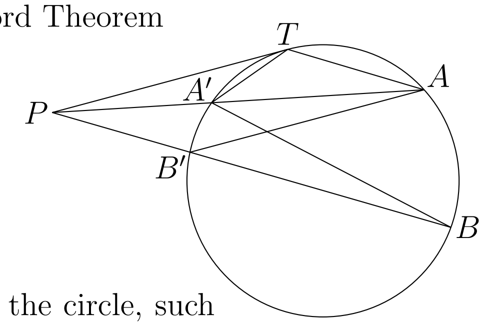


$$
\begin{aligned}
\angle AB'P = \angle AB'B = \angle AA'B = \angle BA'P, &\quad \text{(standing on same arc } AB\text{)}\\
\angle APB' = \angle BPA', &\quad \text{(vertically opposite angles)}\\
\therefore \triangle AB'P \sim \triangle BA'P, &\quad \text{by the AA Rule}\\
\therefore \frac{PA}{PB'} = \frac{PB}{PA'} &\\
\therefore PA \times PA' = PB \times PB'. &
\end{aligned}
$$

Now this result holds for any such pair of points B, B' on the circle, such that BB' passes through P. Consider the case where BB' is the diameter of the circle, let O, R be the centre and radius of the circle, and let $d = OP$. Then one of PB and PB' has length $R + d$ and the other $R - d$, i.e.

$$
\begin{aligned}
PB \times PB' &= (R + d)(R - d)\\
&= R^2 - d^2.
\end{aligned}
$$

Now suppose P is outside the circle, and let T be a point on the circle such that PT is tangent to the circle.



$$
\begin{aligned}
\angle PTA' = \angle A'AT = \angle PAT &\quad \text{by the Tangent-chord Theorem}\\
\angle TPA' = \angle APT, &\quad \text{(same angle)}\\
\therefore \triangle TPA' \sim \triangle PAT, &\quad \text{by the AA Rule}\\
\therefore \frac{PT}{PA'} = \frac{PA}{PT} &\\
\therefore PA \times PA' = PT^2. &
\end{aligned}
$$

Now this result holds for any other pair of points B, B' on the circle, such that BB' passes through P, i.e.

$$PB \times PB' = PT^2 = PA \times PA'.$$

In particular, for the case where BB' is the diameter of the circle, as before, let O, R be the centre and radius of the circle, and let $d = OP$. Then one of PB and PB' has length $d + R$ and the other $d - R$, i.e.

$$
\begin{aligned}
PB \times PB' &= (d + R)(d - R)\\
&= d^2 - R^2.
\end{aligned}
$$

In the case, that P lies on the circle, then exactly one of each pair of points A, A' and B, B' is P, say $P = A' = B'$, in which case, $PA' = PB' = 0$ and hence

$$PA \times PA' = 0 = PB \times PB'.$$

This case occurs precisely when $OP = d = R$, i.e. when

$$d^2 - R^2 = 0 = R^2 - d^2.$$

So finally by assigning a positive direction to one of the segments PA and PA', we see that when P is inside the circle, PA and PA' are oppositely directed so that $PA \times PA'$ is negative, i.e. with this convention,

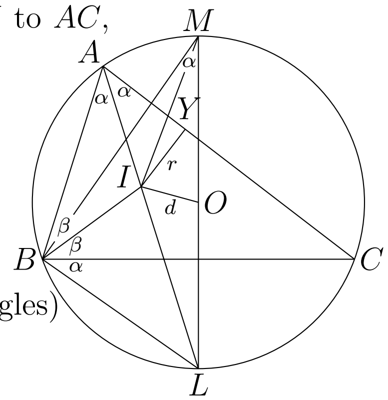
$$PA \times PA' = -(R^2 - d^2) = d^2 - R^2,$$

which is then the same formula in d and R as the case for P outside (or on) the circle. □

Theorem 12.18.4 (Euler). *Let O, R, I, r be the circumcentre, circumradius, incentre, and inradius of $\triangle ABC$ and let $d = OI$. Then*

$$d^2 = R^2 - 2rR.$$

Proof. Let K be the circumcircle of $\triangle ABC$. Produce AI to intersect K again at L. Produce LO to meet K again at M. Drop a perpendicular from I to AC, and let the foot of the perpendicular be Y, so that $IY = r$.
Let $\alpha = \frac{1}{2}\angle A$ and let $\beta = \frac{1}{2}\angle B$. Then



$$\angle BML = \angle BAL = \alpha, \qquad \text{(angles standing on arc } BL\text{)}$$
$$\text{and } \angle LBC = \angle LAC = \alpha, \qquad \text{(angles standing on arc } LC\text{)}$$

Now,

$$\begin{aligned} \angle BIL &= \alpha + \beta, && \text{(ext. angle is sum of int. opp. angles)} \\ &= \angle LBI \end{aligned}$$
$$\therefore \triangle BLI \text{ is isosceles}$$
$$\therefore LI = LB.$$

Thus now we have,

$$\begin{aligned} R^2 - d^2 &= LI \times IA, && \text{(power of point } I \text{ relative to } K\text{)} \\ &= LB \times IA \\ &= LM \cdot \frac{LB}{LM} \cdot \frac{IA}{IY} \cdot IY \\ &= LM \cdot \frac{\frac{LB}{LM}}{\frac{IY}{IA}} \cdot IY \\ &= LM \cdot \frac{\sin\alpha}{\sin\alpha} \cdot IY \\ &= 2R \cdot r \\ \therefore d^2 &= R^2 - 2rR. \end{aligned}$$

□

Exercise Set 12.

1. Prove for any $\triangle ABC$, even if B or C is obtuse, that
$$a = b\cos C + c\cos B.$$
Thus use the Sine Rule to deduce the *addition formula:*
$$\sin(B + C) = \sin B\cos C + \sin C\cos B.$$

2. Prove for $\triangle ABC$,
$$a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B) = 0.$$

3. Prove for $\triangle ABC$, $|ABC| = \dfrac{abc}{4R}$.

4. For $\triangle ABC$, let p and q be the radii of two circles through A, touching BC at B and C, respectively. Prove $pq = R^2$.

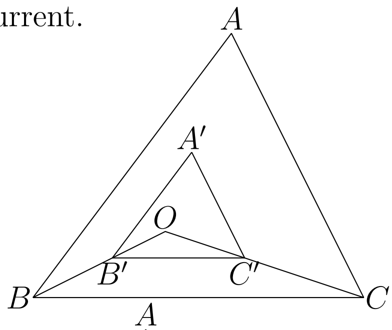
5. If X, Y, Z are the midpoints of sides BC, CA, AB, respectively, of $\triangle ABC$, prove the cevians AX, BY, CZ are concurrent.

   The cevians AX, BY, CZ here, are the *medians* of $\triangle ABC$ and the point at which they concur is the *centroid* or *centre of gravity* of $\triangle ABC$.

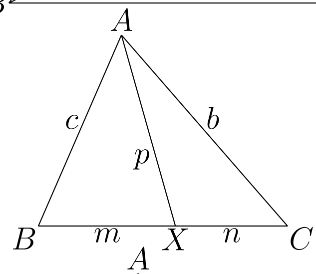
6. Prove cevians perpendicular to the opposite sides are concurrent.

   Such cevians of a triangle are its *altitudes* and the point at which they concur is the *orthocentre*.

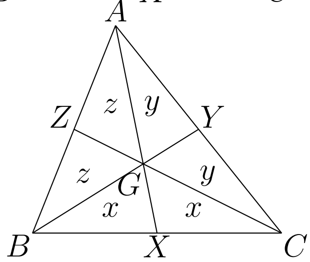
7. Let $\triangle ABC$ and $\triangle A'B'C'$ be non-congruent triangles whose corresponding sides are parallel. Prove the three lines AA', BB' and CC' (extended) are concurrent. Such triangles are said to be **homothetic**.



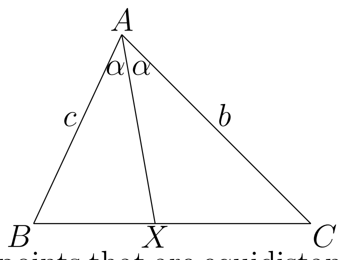
8. Let AX be a cevian of $\triangle ABC$ of length p dividing BC into segments $BX = m$ and $XC = n$. Prove
$$a(p^2 + mn) = b^2m + c^2n.$$
This result is known as **Stewart's Theorem.**



9. Prove that the medians of a triangle dissect the triangle into six smaller triangles of equal area.
   Hint. Suppose in the diagram that x, y, z represent the areas of the smaller triangles they lie in. Start by showing the two triangles marked x have the same area, and similarly for y and z.



10. Prove the medians of a triangle divide each other in the ratio 2 : 1, i.e. the medians of a triangle *trisect* one another.



11. Prove that each (internal) angle bisector of a triangle divides the opposite side into segments proportional in length to the adjacent sides, e.g. if AX is the cevian of $\triangle ABC$ that bisects the angle at A internally, then $BX : XC = c : b$.

12. The angle bisector of the angle between two sides is the locus of points that are equidistant from the sides making the angle. One consequence of this is that any pair of internal angle bisectors of a triangle meet at a point that is equidistant from all three sides of the triangle, and hence in fact the three internal angle bisectors are concurrent.

    The point at which the angle bisectors of a triangle concur is the *incentre* I, the common (perpendicular) distance from I to the three sides is the *inradius* r, and the circle with centre I and radius r thus touches each side tangentially and is called the *incircle* of the triangle.

    Find an alternative proof that the (internal) angle bisectors of a triangle concur, using Ceva's Theorem and the result of the previous problem.

13. Prove the circumcentre and orthocentre of an obtuse-angled triangle lie outside the triangle.

14. Find the ratio of the area of a given triangle to that of a triangle whose sides have the same lengths as the medians of the original triangle.

15. Prove a triangle with two equal medians is isosceles.

16. Prove a triangle with two equal altitudes is isosceles.

17. Suppose that AX, of Exercise 8., is a median of $\triangle ABC$. Find the length of AX in terms of a, b, c.

18. If cevian AX of $\triangle ABC$ bisects $\angle A$, prove
$$AX^2 = bc\left(1 - \left(\frac{a}{b+c}\right)^2\right).$$

19. Find the length of the internal bisector of the right angle in a triangle with sides $3, 4, 5$.

20. Prove the product of two sides of a triangle is equal to the product of the circumdiameter and the altitude on the third side.

    For $\triangle ABC$ sides are labelled a, b, c opposite the vertices A, B, C respectively. Also,
$$\begin{aligned} R &= \text{circumradius of } \triangle ABC \\ r &= \text{inradius of } \triangle ABC \\ s &= \frac{a+b+c}{2} = \text{semiperimeter of } \triangle ABC \end{aligned}$$

21. Let I be the incentre of $\triangle ABC$, and let X, Y, Z be the feet of the perpendiculars dropped from I to sides BC, CA, AB, respectively, and let
$$x = AZ = AY,\ y = BX = BZ,\ z = CX = CY.$$
    Prove
$$x = s - a,\ y = s - b,\ z = s - c.$$

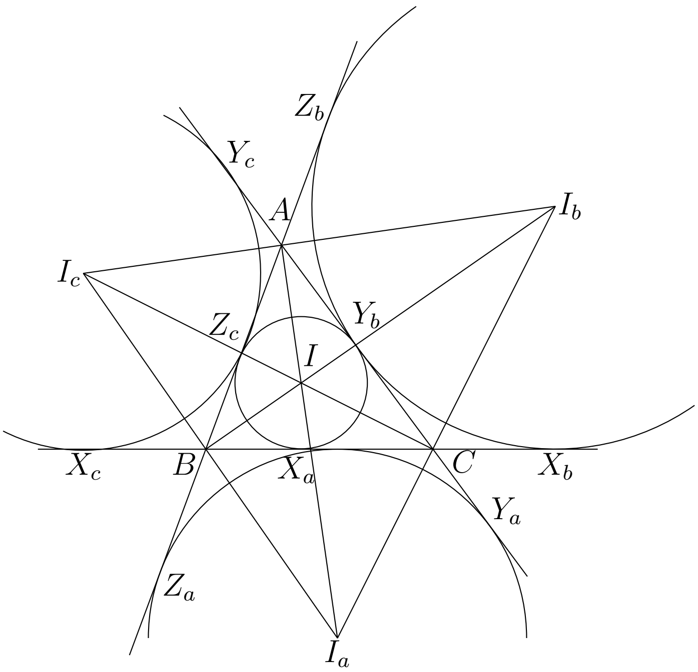
22. Prove $|ABC| = sr$.

23. Prove the external bisectors of two angles of a triangle are concurrent with the internal bisector of the third angle.

24. If three circles with centres A, B, C all touch one another, show their radii are $s-a$, $s-b$, $s-c$, respectively.

25. Prove $abc = 4srR$.

26. If X, Y, Z are the feet of the perpendiculars dropped from the incentre I of $\triangle ABC$, as in Exercise 21., prove the cevians AX, BY, CZ concur. Their point of concurrence is called the **Gergonne point** of $\triangle ABC$.



If one produces the sides of a triangle beyond its vertices, external angles are formed at the vertices; so one can construct **external angle bisectors**. Each external angle bisector is perpendicular to the corresponding *internal angle bisector*. Let the intersections of the external angle bisectors of $\triangle ABC$ be I_a, I_b and I_c, labelled according to their being opposite vertices A, B and C, respectively.

27. Prove $\triangle ABC$ is the orthic triangle of $\triangle I_a I_b I_c$.
    Note. If K, L, M are the feet of the altitudes of $\triangle DEF$ then $\triangle KLM$ is the **orthic triangle** of $\triangle DEF$.

28. Prove that each of I_a, I_b and I_c, are equidistant from the sides of $\triangle ABC$, and let these common distances be r_a, r_b and r_c, respectively.
    Thus, r_a, r_b, r_c are **exradii** of $\triangle ABC$, i.e. radii of **excircles** of $\triangle ABC$ (circles touching sides of $\triangle ABC$ externally), with respective centres (**excentres**) I_a, I_b, I_c.
    Thus prove $|ABC| = (s-a)r_a = (s-b)r_b = (s-c)r_c$.

29. Prove $\dfrac{1}{r_a} + \dfrac{1}{r_b} + \dfrac{1}{r_c} = \dfrac{1}{r}$.

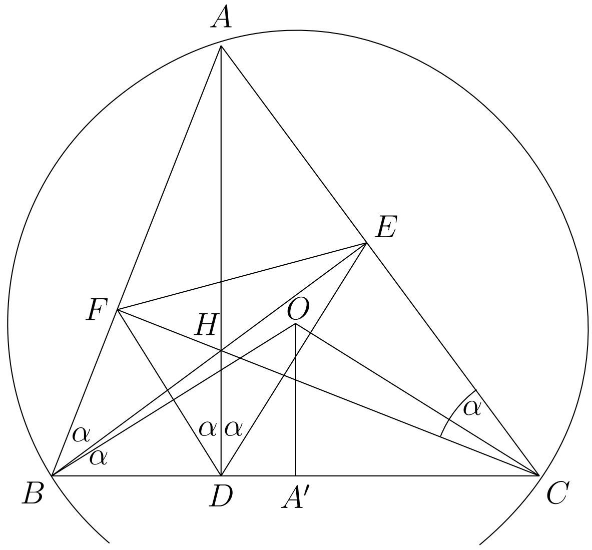
Part (ii) of the following theorem is known as the **Steiner-Lehmus Theorem**.

Theorem. *(i) If a triangle has two different angles then the smaller angle has the longer internal bisector.*

(ii) If a triangle has two equal-length (internal) angle bisectors then it is isosceles.

30. Let $\angle B = 12^\circ$ and $\angle C = 132^\circ$ of $\triangle ABC$. Without using trigonometric functions, compare the lengths of the external angle bisectors from B and C to their respective opposite sides (produced).

31. Prove the orthocentre of an acute-angled triangle is the incentre of its orthic triangle.
Hint. The diagram is of $\triangle ABC$ with circumcentre at O, feet of altitudes at D, E and F, and orthocentre at H. Also A' is the foot of the perpendicular dropped from O to BC. Start by showing the angles marked α in the diagram are all equal to $90^\circ - \angle A$, so that AD bisects $\angle EDF$.



32. Prove $\triangle AEF \sim \triangle DBF \sim \triangle DEC \sim \triangle ABC$.

33. Prove $\angle HAO = |\angle B - \angle C|$.

34. What is the minimum value that the power of a point can have, relative to a given circle of radius R? Which point has this power?

35. What is the locus of points of constant power relative to a given circle of radius R?

36. If the power of a point has the positive value t^2, interpret the length t geometrically.

37. Given PT and PU are tangents from P to two concentric circles, with T on the smaller circle, and PT meets the larger circle at Q, prove

$$PT^2 - PU^2 = QT^2.$$

38. Prove the circumradius of a triangle is at least twice the inradius.

39. Express in terms of the inradius r and circumradius R, the power of the incentre relative to the circumcircle of a triangle.

40. (AIMO 2013 Q8) A circle K meets equilateral triangle ABC in points D, E, F, G, H, I (in that order), where D, E are on CA, F, G are on AB and H, I are on BC.
Also, $AE = 4$, $ED = 26$, $DC = 2$, $FG = 14$ and a circle with diameter HI has area πb. Find b.

41. Prove that the notation of directed segments enables us to express Stewart's Theorem in the following symmetrical form.

If P, A, B, C are four points of which the last three are *collinear*, then

$$PA^2 \cdot BC + PB^2 \cdot CA + PC^2 \cdot AB + BC \cdot CA \cdot AB = 0.$$

Directed segment convention. The product of two directed segments on one line is regarded as being positive or negative according as whether the directions are aligned or opposite.

42. A line through the centroid G of $\triangle ABC$ intersects the sides of the triangle in points X, Y, Z. Prove that

$$\frac{1}{GX} + \frac{1}{GY} + \frac{1}{GZ} = 0,$$

using the directed segment convention.

43. How far away is the horizon as seen from the top of a 2 km high mountain? (Assume the earth is a sphere of radius 6399 km.)