CHAPTER 1

Sets and Logic

In this chapter, we establish the *notation* and fundamental *notions* of *sets*, and discuss the correspondence of *sets* with *logic*.

1.1 Sets and membership

A *set* is a "assemblage"* of objects. These objects are called the *elements* or *members* of the *set,* where the objects can be anything: numbers, people, other sets, etc. E.g., 12 is an element of the *set of even integers*.

If x is *an element* of A, then it is also said that x *belongs to* A, or that x *is in* A, or that A *contains* x. In this case, we write $x \in A$. The symbol $\in$ was introduced by Peano in 1889, and is derived from the Greek letter epsilon ϵ (the letter in the Greek alphabet corresponding to the Roman letter 'e' which is of course the initial letter of *element*).

We may also say A *contains* x, and write $A \ni x$. If x *is not in* A, we write: $x \notin A$.

1.2 Specifying sets

The simplest way to describe a *set* is by *enumeration,* i.e. by listing its elements explicitly between *curly braces*. Thus $\{1, 2\}$ denotes the set whose only elements are 1 and 2. Note the following two properties of sets:

- Order of elements is immaterial, e.g. $\{1, 2\} = \{2, 1\}$.
- Repetition (multiplicity) of elements is irrelevant, e.g. $\{1, 1, 2, 2\} = \{1, 2, 2, 2\} = \{1, 2\}$.

A *set* can have *no elements*. In this case we say the set is *empty* and denote the *empty set* by $\{\}$. We also use the term *null set* and the notation $\emptyset$ for the *empty set*.

The alternative way to represent a set is with *set-builder notation,* which has the form

$$\{\textit{pattern} \mid \textit{condition(s)}\}.$$

Typically, we write $\{x \mid P(x)\}$, or $\{x : P(x)\}$, to denote *the set containing all objects* x *such that the condition or property* P *holds for* x, e.g. we may write

$$\{x \mid x \text{ is a prime}\},$$

literally read as:

the set of all x *such that* x *is a prime*,

which, in this case, we could say more succinctly as: *the set of prime numbers*. Usually, we read the symbol '|' as *'such that'*. The *pattern* may also be an expression, e.g.

$$\{p^2 \mid p \text{ is a prime}\}$$

is the *set of all numbers that are the squares of prime numbers*.

*We might have called a *set* a "collection" of objects, but, a *collection* already means something else; a **collection** is a *set* whose members are themselves sets, e.g. $\big\{\{1\}, \{1, 2\}\big\}$ is a collection.

1.3 Special sets

Some sets turn up so often that we have special symbols for them in a special typeface known as *Blackboard Bold:*

$$
\begin{aligned}
\mathbb{N} &= \{1, 2, 3, \dots\} &&= \text{(the set of) } \textit{Natural Numbers}\\
\mathbb{Z} &= \{\dots, -2, -1, 0, 1, 2, 3, \dots\} &&= \text{(the set of) } \textit{Integers}\\
\mathbb{Q} &= \left\{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ with } b \neq 0\right\} &&= \text{(the set of) } \textit{Rational Numbers}\\
\mathbb{R} & &&= \text{(the set of) } \textit{Real Numbers}
\end{aligned}
$$

The last set contains all numbers that can be represented on the *number line* including the *rational numbers* and numbers such as $\sqrt{2}, \sqrt{3}, \pi, \dots$ that are termed *irrational.*

1.4 Variants of set-builder notation

Set builder notation may also be used as follows:

- $\{x \in A \mid P(x)\}$ denotes the set of all x that are already in A such that x has the property P, e.g. $\{x \in \mathbb{Z} \mid x \text{ is even}\}$ is *the set of all even integers.*
- $\{f(x) \mid x \in A\}$ denotes the set of all objects with pattern $f(x)$ such that x is in A. We saw this form above, in the definition of the *rational numbers* $\mathbb{Q}$. For a simpler example, consider: $\{2x \mid x \in \mathbb{Z}\}$ is another way of specifying *the set of all even integers.*
- $\{f(x) \mid P(x)\}$ is the most general form of set builder notation, e.g. above we saw: $\{p^2 \mid p \text{ is a prime}\}$, *the set of squared prime numbers.*

1.5 Subsets and supersets

Given two sets A and B, we say that A is a *subset* of B, and write $A \subseteq B$, if every element of A is also an element of B. Notice that in particular, B is a subset of itself. If a subset A of B is *not equal* to B, we say A is a *strict subset* of B or that A is a *proper subset* of B.

If A is a subset of B, then one can also say that B is a *superset* of A, and write: $B \supseteq A$. We also say that A *is contained in* B, or that B *contains* A.

Note that two sets A, B are equal, written $A = B$, if and only if both $A \subseteq B$ and $B \subseteq A$. Usually when trying to prove that two sets are equal, one shows each set is contained in the other.

Let $\mathbb{P} = \{p \in \mathbb{N} \mid p \text{ is prime}\}$, then we have the following sequence of inclusions, i.e. a sequence of sets for which each set is a subset of the next:

$$\mathbb{P} \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

1.6 Universal sets and complements

Often we consider all sets as being subsets of some given *universal set.* E.g., if we are investigating properties of the real numbers $\mathbb{R}$ (and subsets of $\mathbb{R}$), then we may take $\mathbb{R}$ as our *universal set.*

Given a universal set U and a subset A of U, we may define the *complement* of A (in U) as

$$A^{\mathrm{c}} = \{x \in U \mid x \notin A\}.$$

In other words, A^{c} ('A-complement') is the set of all elements of U which are not elements of A. The notations A' and $\overline{A}$ are also commonly used to represent the *complement of* A. Thus, the complement E^{c} of the set $E = \{2x \mid x \in \mathbb{Z}\}$ (the set of all even integers) in $\mathbb{Z}$, is the *set of all odd integers*, while the complement of E in $\mathbb{R}$ is the set of all real numbers that are either odd integers or not integers at all.

1.7 Unions, intersections, and relative complements

Given two sets A and B, their *union*, written $A \cup B$, is the set consisting of all objects which are elements of A *or* of B (or of both).

The *intersection* of A and B, written $A \cap B$, is the set of all objects which are both in A *and* in B.

Finally, the *relative complement* of B *relative to* A, also known as the *set(-theoretic) difference* of A and B, is the set of all objects that belong to A but not to B. It is written as $A \setminus B$.

Formally, these sets are:

$$\begin{aligned} A \cup B &= \{x \mid (x \in A) \text{ or } (x \in B)\}, \\ A \cap B &= \{x \mid (x \in A) \text{ and } (x \in B)\}, \\ A \setminus B &= \{x \in A \mid x \notin B\}. \end{aligned}$$

The *(absolute) complement* A^{c} of a set A (in a universal set U) using the *set difference* notation is $U \setminus A$.

1.8 De Morgan's Laws

The following statements, known as *de Morgan's Laws,* are true for any sets A and B. Each is easy to prove using Venn diagrams.

$$\begin{aligned} (A \cup B)^{\mathrm{c}} &= A^{\mathrm{c}} \cap B^{\mathrm{c}} \\ (A \cap B)^{\mathrm{c}} &= A^{\mathrm{c}} \cup B^{\mathrm{c}} \end{aligned}$$

Note that when we draw a Venn diagram that is supposed to represent a general situation each set should be drawn to intersect each other set (this represents a general situation, since it can still happen that any region may actually be empty). Thus, a Venn diagram with two sets A and B should be drawn as two intersecting circles in a rectangle representing the *universal set* U.

1.9 Symbols Summary

$x \in A$	means that x is an **element** of A.
$A \cup B$	is the **union** of A and B, i.e. $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ (in mathematics "or" means "and/or").
$A \cap B$	is the **intersection** of A and B, i.e. $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.
$A \subseteq B$	means that A is a **subset** of B, i.e. if $x \in A$ then $x \in B$.
$A \subset B$	means that A is a **proper subset** of B, i.e. $A \subseteq B$ and $A \neq B$.
$A \setminus B$	(or $A - B$) is A **take** B, i.e. the set $\{x \in A \mid x \notin B\}$.
$\{\}, \emptyset$	is the **empty set**, i.e. the set with no elements. Note that $\emptyset$ is a subset of every set.
$A^c, A', \overline{A}$	is the **complement** of A, i.e. $A' = \{x \mid x \notin A\}$. Here x belongs to some "universal set" which should be clear from the context.

1.10 The connection between Set Theory and Logic

Logic deals with *statements* that are either `true` or `false`, whereas Set Theory deals with *elements* of sets – a given *element* can either be *in* a given set or *not in* the set. Let A, B represent *sets*. Also, let p, q represent *statements* and p' represents the **negation** of p (if p is `true` then p' is `false`, and vice-versa).

A right arrow ($\rightarrow$) denotes **implies**. If $p \rightarrow q$, then, when p is `true` so is q.

A double-arrow ($\leftrightarrow$) is the corresponding symbol for equals; it denotes its operands are **logically equivalent**. If $p \leftrightarrow q$ then, when p is `true`, so is q, and when p is `false`, so is q).

A way of viewing the connection between Set Theory and Logic: is to say that the *statements* in Logic tell us which regions of a given *Universal set* are *non-empty*. With this view, one can often convert a Set Theory statement to a Logic one by sticking '$x \in$' in front of it, e.g.

$$\begin{aligned} A \cup B &\longrightarrow x \in (A \cup B) \\ &\leftrightarrow x \in A \text{ or } x \in B \end{aligned}$$

Now, $x \in A$ and $x \in B$ are examples of statements p and q, respectively.

We give a few examples of this correspondence between Set Theory and Logic:

Set Theory	Logic
$A \cup B$	p or q
$A \cap B$	p and q
$A = B$	$p \leftrightarrow q$
$A \subseteq B$	$p \rightarrow q$
$(A \cup B)' = A' \cap B'$	$(p \text{ or } q)' \leftrightarrow p' \text{ and } q'$
$(A \cap B)' = A' \cup B'$	$(p \text{ and } q)' \leftrightarrow p' \text{ or } q'$

The statement '$p \rightarrow q$' has particular importance. From the Set Theory perspective, we are saying that $A \cap B' = \emptyset$ or equivalently that the *Universal set* is $A' \cup B$, so that we have the following logical equivalence:

$$p \rightarrow q \quad \leftrightarrow \quad p' \text{ or } q.$$

In Logic it is customary to use $\vee$ (vee) for 'or', $\wedge$ (wedge) for 'and' and $\neg$ for **negation** (i.e. we write $\neg p$ for p'). So the above statement may also be written as it appears in the first exercise.

1.11 Logic with Quantifiers

The following are known as the **universal quantifier** and **existential quantifier**, respectively.

$\forall$ (which is an inverted **A**) means "for **A**ll".

$\exists$ (a back-to-front **E**) means "there **E**xists".

If $P(x)$ represents a statement that depends on x, then the following are *logical equivalences:*

$$\exists x P(x) \leftrightarrow \neg\forall x\big(\neg P(x)\big)$$
$$\forall x P(x) \leftrightarrow \neg\exists x\big(\neg P(x)\big).$$

Sometimes we write the quantifiers at the back of an expression rather than at the front. The meaning is the same, but a trailing '$\exists x$' reads better as: '*for some* x'.

Exercise Set 1.

1. Use a truth table to prove that $p \to q$ is logically equivalent to $\neg p \vee q$, i.e. show
$$(p \to q) \leftrightarrow (\neg p \vee q)$$
is a **tautology** (i.e. has value `true` for all possible inputs of p and q).

2. Use 1. to show $p \to q$ is logically equivalent to its contrapositive $\neg q \to \neg p$.

3. Write the statement,

   In every class there is at least one student who gets everything right.

   symbolically using quantifiers.
   Hint. For example, let $R(c, s, p)$ represent "Student s in class c gets problem p correct."

   Then write the *negation* of this statement as an *existential* statement (i.e. beginning with $\exists$), symbolically, and then translate that statement back into English.

4. Noting that we often add "s.t." (abbreviating "*such that*") to improve the readability of a clause following $\exists$, rewrite
$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ s.t. } y = \cos(x)$$
so that the first quantifier in the sentence is $\exists$.