

Vectors

13.1 Introduction and notation

Definition 13.1.1. A **vector** is a quantity that has both *magnitude* and *direction*, e.g. wind described by speed and direction, such as, 20 km/h north-east, force, displacement. Essentially, a *vector* is a directed line segment. Geometrically, vectors are represented as *arrows*; the tail of the arrow is its initial point and the tip is its terminal point.

In \mathbb{R}^3 (3-space), the directed line segment from the origin $O(0, 0, 0)$ to the point $P(x_1, y_1, z_1)$ is the **position vector** of the point $P(x_1, y_1, z_1)$.

The **vector**

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

denotes *any* directed line segment *parallel* (and pointing in the same direction) and having the same *length* as the *position vector* of the point (x_1, y_1, z_1) . In this way, a *vector* has *magnitude* (length) and *direction*, but not *position*.

Many modern textbooks use the more compact notation $\langle x_1, y_1, z_1 \rangle$ or the same notation as for the point (x_1, y_1, z_1) for the vector

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix},$$

but the vertical notation has particular advantages when adding vectors or calculating *scalar* (*dot*) products (which we define later), since the components being added or multiplied are side-by-side.

For two points, A and B in space, the *vector* parallel to the directed line segment from A to B may be represented by \overrightarrow{AB} . The *position vector* of P above, is \overrightarrow{OP} . Another notation is to use a lower case letter with a tilde underneath; textbooks commonly instead embolden the letter; in these notes, we will do both together, e.g.

$$\underline{\mathbf{u}} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

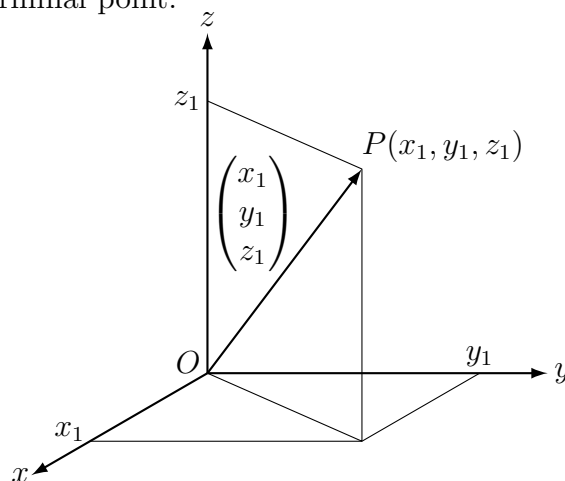
The above also demonstrates a convention; if a certain letter is used to represent a vector (in this case, u), then its x -, y - and z - components are that letter with subscripts 1, 2 and 3, respectively.

A lowercase letter without such adornments typically represents a quantity with magnitude, and except possibly for sign, not direction. Such non-vectors are called **scalars**.



It will become important to distinguish *scalar* quantities from *vectors*, and hence important to remember the tildes!

Two vectors of the same length and direction (but not necessarily position) are **equivalent** (i.e. **equal**).



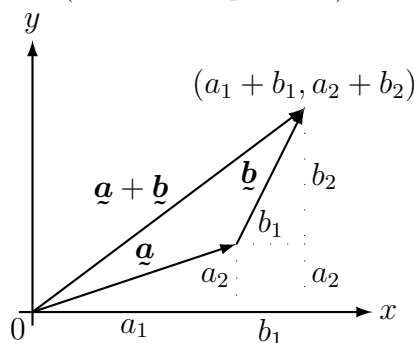
13.2 Vector addition

To add two vectors, we add their corresponding components:

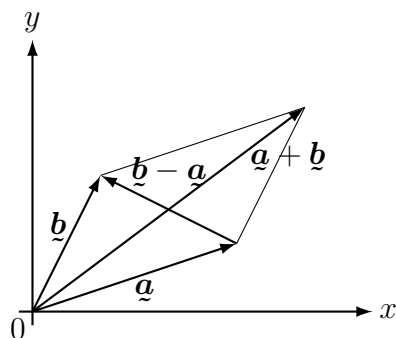
$$\begin{aligned}\underline{\mathbf{a}} + \underline{\mathbf{b}} &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}.\end{aligned}$$

We have demonstrated *vector addition* for \mathbb{R}^2 vectors, but the rule is similar with \mathbb{R}^3 vectors (three components), or more generally, with \mathbb{R}^n vectors (with n components).

Geometrically, we can add vectors by placing them head to tail as shown. This is the **Triangle Law**.



The **Parallelogram Law** is another approach, where we place the tails of the vectors ($\underline{\mathbf{a}}, \underline{\mathbf{b}}$ here) at the origin. Then the vectors make two sides of a parallelogram. The fourth vertex of this parallelogram is at the tip of the sum vector ($\underline{\mathbf{a}} + \underline{\mathbf{b}}$ here). The other diagonal of the parallelogram gives the difference vector $\underline{\mathbf{b}} - \underline{\mathbf{a}}$. Also notice that the sum of vectors around a closed path is $\underline{\mathbf{0}}$ (the zero vector, which we will describe later).



13.3 Scalar multiplication

For $k \in \mathbb{R}$, $\underline{\mathbf{a}} \in \mathbb{R}^n$,

$$k\underline{\mathbf{a}} = k \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} ka_1 \\ ka_2 \\ \vdots \\ ka_n \end{pmatrix}$$

is a vector in the same direction as $\underline{\mathbf{a}}$, but of $|k|$ times the length of $\underline{\mathbf{a}}$, so that $k\underline{\mathbf{a}}$ is **parallel** to $\underline{\mathbf{a}}$ (written: $k\underline{\mathbf{a}} \parallel \underline{\mathbf{a}}$).

If $k < 0$ then $k\underline{\mathbf{a}}$ is oppositely directed to $\underline{\mathbf{a}}$ (and we can impart this extra information by saying $k\underline{\mathbf{a}}$ is **antiparallel** to $\underline{\mathbf{a}}$).

13.4 Length of a vector

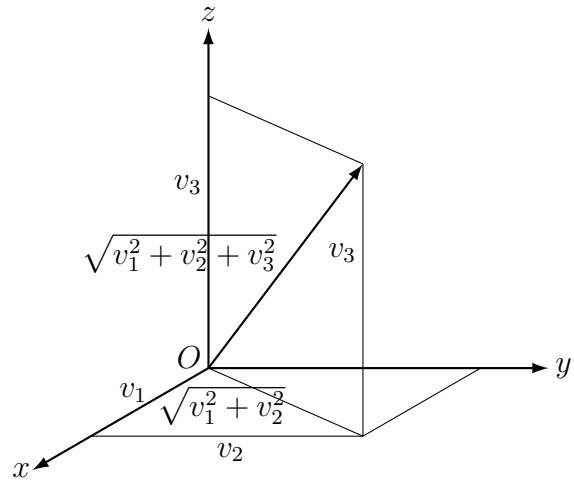
In general, if


$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

then the **length** of \mathbf{v} is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

The diagram shows the case for $n = 3$; and if you look in the (x, y) -plane also the $n = 2$ case, from which the extension to $n = 3$ is quite natural.



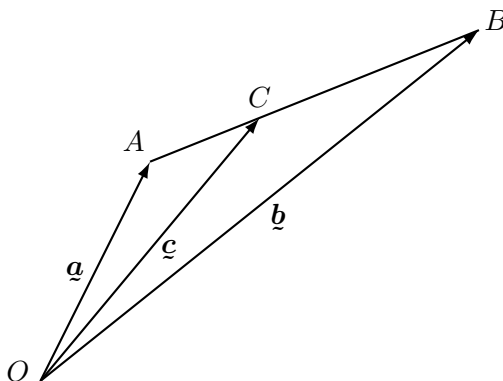
 We write the length of a vector with “double bar” notation ($\|\cdot\|$). Most texts use “single bar” notation, so that it looks like *absolute value*. Sometimes, however, it is helpful to be able to distinguish the two notations, as with the next lemma.

Lemma 13.4.1. $\|k\mathbf{v}\| = |k| \|\mathbf{v}\|$.

The lemma says that the effect of multiplying a vector by a scalar k is to increase its length by a factor $|k|$.

Example 13.4.2. Given C is a point on the line segment AB , twice as far from B as from A , write the position vector $\mathbf{c} = \overrightarrow{OC}$ of C in terms of $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$.

Solution.



$$\begin{aligned} \mathbf{a} + \overrightarrow{AB} &= \mathbf{b} \\ \therefore \overrightarrow{AB} &= \mathbf{b} - \mathbf{a} \\ \overrightarrow{AC} &= \frac{1}{3}\overrightarrow{AB} \\ &= \frac{1}{3}(\mathbf{b} - \mathbf{a}) \\ \mathbf{c} &= \mathbf{a} + \overrightarrow{AC} \\ &= \mathbf{a} + \frac{1}{3}(\mathbf{b} - \mathbf{a}) \\ &= \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}. \end{aligned}$$

Exercise 13.4.3. For triangle ABC , let the position vectors of the vertices be $\mathbf{a}, \mathbf{b}, \mathbf{c}$, respectively. Find expressions in terms of these vectors for the midpoints of the sides, and the centroid G . Also, show that the medians trisect one another, in the sense that, if X is the midpoint of BC , then $AG : GX = 2 : 1$.

13.5 Properties

Suppose $\underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}} \in \mathbb{R}^n$ and $k, \ell \in \mathbb{R}$. Then the following properties hold.

1. $\underline{\mathbf{a}} + \underline{\mathbf{b}} = \underline{\mathbf{b}} + \underline{\mathbf{a}}$. (commutativity of vector +)

2. $(\underline{\mathbf{a}} + \underline{\mathbf{b}}) + \underline{\mathbf{c}} = \underline{\mathbf{a}} + (\underline{\mathbf{b}} + \underline{\mathbf{c}})$. (associativity of vector +)

3. There is a vector $\underline{\mathbf{0}}$ such that

$$\underline{\mathbf{0}} + \underline{\mathbf{a}} = \underline{\mathbf{a}} + \underline{\mathbf{0}} = \underline{\mathbf{a}}.$$

In \mathbb{R}^n ,

$$\underline{\mathbf{0}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and is called the **zero vector**, as opposed to the number 0 which is a **scalar**.

(existence of + **identity**)

4. For every $\underline{\mathbf{a}}$ there is a vector $\underline{\mathbf{b}}$ ($= -\underline{\mathbf{a}}$) such that

$$\underline{\mathbf{b}} + \underline{\mathbf{a}} = \underline{\mathbf{0}} = \underline{\mathbf{a}} + \underline{\mathbf{b}}.$$

Here

$$\underline{\mathbf{b}} = -\underline{\mathbf{a}} = \begin{pmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_n \end{pmatrix}.$$

(existence of + **inverses**)

5. $k(\underline{\mathbf{a}} + \underline{\mathbf{b}}) = k\underline{\mathbf{a}} + k\underline{\mathbf{b}}$. (distribution)

6. $(k + \ell)\underline{\mathbf{a}} = k\underline{\mathbf{a}} + \ell\underline{\mathbf{a}}$. (distribution)

7. $k(\ell\underline{\mathbf{a}}) = (k\ell)\underline{\mathbf{a}}$.

8. $1\underline{\mathbf{a}} = \underline{\mathbf{a}}$.

13.6 Special vectors

Definition 13.6.1. A **unit vector** is a vector of length 1. In general, a *unit vector* may be formed from any non-zero vector by dividing through by its length. We add a *caret* above the symbol for a vector to indicate “the unit vector formed from” that vector; thus

$$\hat{\underline{\mathbf{a}}} = \frac{1}{\|\underline{\mathbf{a}}\|} \underline{\mathbf{a}}.$$

The *unit vectors* in \mathbb{R}^3 , in the x -, y - and z - directions, respectively, are

$$\underline{\mathbf{i}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{\mathbf{j}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \underline{\mathbf{k}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Lemma 13.6.2. $\hat{\mathbf{v}} = (c\mathbf{v})^\wedge$, for any $c > 0$.

The usefulness of the above lemma is that it tells us that we can clear ugly common factors first. This may greatly simplify the process of finding unit vectors.

Example 13.6.3. 1. Find the unit vector for $\mathbf{v} = \begin{pmatrix} \frac{1}{12} \\ \frac{1}{15} \\ \frac{1}{20} \end{pmatrix}$.

Solution.

$$\hat{\mathbf{v}} = (60\mathbf{v})^\wedge = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}^\wedge = \frac{1}{\sqrt{50}} \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} = \frac{1}{5\sqrt{2}} \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}.$$

2. Find the unit vector for $\begin{pmatrix} \frac{1}{2\sqrt{2}} \\ \frac{1}{-2\sqrt{2}} \end{pmatrix}$.

Solution.

$$\begin{pmatrix} \frac{1}{2\sqrt{2}} \\ \frac{1}{-2\sqrt{2}} \end{pmatrix}^\wedge = \begin{pmatrix} 1 \\ -1 \end{pmatrix}^\wedge = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

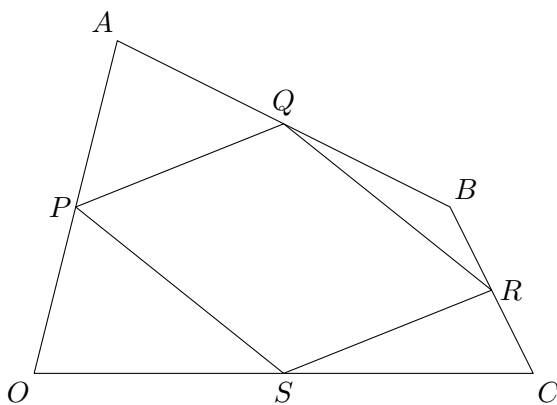
3. Find the unit vector for $\begin{pmatrix} 123 \\ 246 \\ 246 \end{pmatrix}$.

Solution.

$$\begin{pmatrix} 123 \\ 246 \\ 246 \end{pmatrix}^\wedge = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}^\wedge = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

Example 13.6.4. Prove that the midpoints of the sides of a quadrilateral form a parallelogram.

Solution.



Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$. Then

$$\begin{aligned} \overrightarrow{PA} &= \frac{1}{2}\mathbf{a} \\ \overrightarrow{AQ} &= \frac{1}{2}\overrightarrow{AB} \\ &= \frac{1}{2}(\mathbf{b} - \mathbf{a}) \\ \overrightarrow{PQ} &= \overrightarrow{PA} + \overrightarrow{AQ} \\ &= \frac{1}{2}\mathbf{a} - \frac{1}{2}(\mathbf{b} - \mathbf{a}) \\ &= \frac{1}{2}\mathbf{b} \end{aligned}$$

Similarly, $\overrightarrow{SR} = \frac{1}{2}\mathbf{b}$, and so $\overrightarrow{PQ} = \overrightarrow{SR}$.

Similarly, $\overrightarrow{PS} = \overrightarrow{QR}$.

Hence, $PQRS$ is a parallelogram.

13.7 The scalar product or dot product

Definition 13.7.1. The scalar product or dot product of two vectors $\underline{a}, \underline{b} \in \mathbb{R}^n$ is defined by

$$\begin{aligned}\underline{a} \cdot \underline{b} &= \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \\ &= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.\end{aligned}$$

Note that the result is a number, i.e. a *scalar* quantity.

13.7.1 Properties of the dot product

If $\underline{a}, \underline{b}, \underline{c} \in \mathbb{R}^n$ and $k \in \mathbb{R}$, then

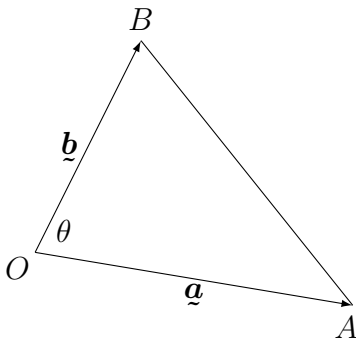
1. $\underline{a} \cdot \underline{a} = a_1^2 + a_2^2 + \cdots + a_n^2 = \|\underline{a}\|^2$.
2. $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$. (commutativity)
3. $\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$. (distribution of \cdot over $+$ for vectors)
4. $(k\underline{a}) \cdot \underline{b} = k(\underline{a} \cdot \underline{b}) = \underline{a} \cdot (k\underline{b})$.
5. $\underline{0} \cdot \underline{a} = 0$.

Note that on the left there is the zero vector and on the right the zero scalar.

All these properties are straightforward to prove directly from the definition.

Theorem 13.7.1.1. $\underline{a} \cdot \underline{b} = \|\underline{a}\| \|\underline{b}\| \cos \theta$, where θ is the angle between the vectors \underline{a} and \underline{b} .

Proof. Consider a triangle OAB .



By the Cosine rule,

$$\begin{aligned}|AB|^2 &= |OA|^2 + |OB|^2 - 2|OA||OB|\cos\theta \\ \|\underline{b} - \underline{a}\|^2 &= \|\underline{a}\|^2 + \|\underline{b}\|^2 - 2\|\underline{a}\|\|\underline{b}\|\cos\theta \\ (\underline{b} - \underline{a}) \cdot (\underline{b} - \underline{a}) &= \underline{a} \cdot \underline{a} + \underline{b} \cdot \underline{b} - 2\|\underline{a}\|\|\underline{b}\|\cos\theta \\ \underline{b} \cdot \underline{b} - \underline{a} \cdot \underline{b} - \underline{b} \cdot \underline{a} + \underline{a} \cdot \underline{a} &= \underline{a} \cdot \underline{a} + \underline{b} \cdot \underline{b} - 2\|\underline{a}\|\|\underline{b}\|\cos\theta \\ -2\underline{a} \cdot \underline{b} &= -2\|\underline{a}\|\|\underline{b}\|\cos\theta \\ \underline{a} \cdot \underline{b} &= \|\underline{a}\|\|\underline{b}\|\cos\theta.\end{aligned}$$

□

13.7.2 Further properties of the dot product

6. $|\underline{a} \cdot \underline{b}| = \|\underline{a}\| \|\underline{b}\|$ if and only if $\underline{a} \parallel \underline{b}$.
7. $\underline{a} \cdot \underline{b} = 0$ if and only if $\underline{a} \perp \underline{b}$.