The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

1996 Senior Mathematics Contest Problems

1. Let K be a semicircle with diameter AB. Let D be a point such that $AB = AD$ and AD intersects K at E. Let F be the point on the chord AE such that $DE = EF$. Let BF extended meet K at C. Show that $\angle BAE = 2\angle EAC$.

2. Find all functions $f(x)$ which are defined for all real numbers x, take real numbers as values and satisfy the equation

$$f(u+v)f(u-v) = 2u + f(u^2 - v^2)$$

for all real numbers u and v.

3. Let x be a non-zero real number such that $x + \dfrac{1}{x} \in \mathbb{Z}$. Prove $x^n + \dfrac{1}{x^n} \in \mathbb{Z}$, for all $n \in \mathbb{N}$.

4. The sequence $a_0, a_1, a_2, \ldots, a_{1997}$ has the properties:

   (i) $0 \leq a_n \leq 1$ for all $0 \leq n \leq 1997$,

   (ii) $a_n \geq \dfrac{a_{n-1} + a_{n+1}}{2}$ for all $1 \leq n \leq 1996$.

   (a) Prove that $a_{1997} - a_{1996} \leq \dfrac{1}{1997}$.

   (b) Find a sequence satisfying (i) and (ii) such that $a_{1997} - a_{1996} = \dfrac{1}{1997}$.

5. Let ABC be an acute-angled triangle with $\angle ACB = 60^\circ$. Let h_a be an altitude through A and let h_b be an altitude through B. Prove that the circumcentre of $\triangle ABC$ lies on the bisector of one of the four angles formed by h_a and h_b.