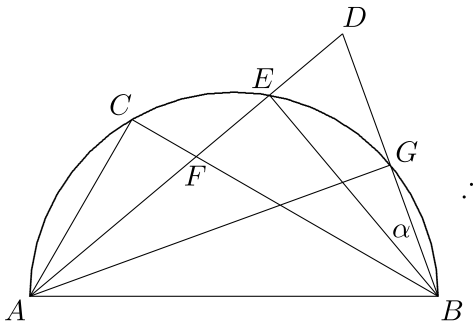
The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

1996 Senior Mathematics Contest Problems with Most Solutions

1. Let K be a semicircle with diameter AB. Let D be a point such that $AB = AD$ and AD intersects K at E. Let F be the point on the chord AE such that $DE = EF$. Let BF extended meet K at C. Show that $\angle BAE = 2\angle EAC$.

   Solution. Let G be the point where DB meets K, and let $\alpha = \angle DBE$.



$$
\begin{array}{rll}
EF = ED, & & \text{(given)} \\
90^\circ = \angle AEB = \angle FEB = \angle DEB, & & \angle AEB \text{ is angle in semicircle,} \\
& & \angle FED \text{ is a straight angle} \\
EB \text{ is common} & & \\
\therefore \triangle FEB \cong \triangle DEB, & & \text{by the SAS Rule} \\
\therefore \angle FBE = \angle DBE = \alpha & & \\
\angle EFB = \angle CFA, & & \text{(opposite angles)} \\
90^\circ = \angle ACB = \angle ACF = \angle FEB, & & \angle ACB \text{ is angle in semicircle} \\
\therefore \triangle ACF \sim \triangle BEF, & & \text{by the AA Rule} \\
\therefore \angle FAC = \angle FBE = \alpha & & \\
90^\circ = \angle BGA = \angle DGA, & & \angle BGA \text{ is angle in semicircle,} \\
& & \angle BGD \text{ is a straight angle} \\
\therefore 90^\circ = \angle DGA = \angle DEB & & \\
\angle ADG = \angle BDE & & \\
\triangle DGA \sim \triangle DEB, & & \text{by the AA Rule} \\
\therefore \angle DAG = \angle DBE = \alpha & & \\
90^\circ = \angle DGA = \angle BGA & & \\
AD = AB, & & \text{(given)} \\
AG \text{ is common} & & \\
\therefore \triangle DGA \cong \triangle BGA, & & \text{by the RHS Rule} \\
\therefore \angle DAG = \angle BAG = \alpha & & \\
\therefore \angle BAE = \angle BAG + \angle DAG & & \\
= 2\alpha & & \\
= 2\angle FAC = 2\angle EAC & &
\end{array}
$$

2. Find all functions $f(x)$ which are defined for all real numbers x, take real numbers as values and satisfy the equation

$$f(u+v)f(u-v) = 2u + f(u^2 - v^2)$$

   for all real numbers u and v.

Solution. Putting $u = v = 0$ we obtain

$$\begin{aligned} f(0)^2 &= 0 + f(0) \\ f(0)^2 - f(0) &= 0 \\ f(0)\big(f(0) - 1\big) &= 0 \end{aligned}$$

so that $f(0) = 0$ or $f(0) = 1$.
Suppose $f(0) = 0$ and put $u = v = x/2 \in \mathbb{R}$. Then

$$\begin{aligned} f(2u)f(0) &= 2u + f(0) \\ 0 &= x, \qquad\qquad \text{(contradiction, e.g. for } x = 1\text{).} \end{aligned}$$

Thus, $f(0) \neq 0$. Hence, if functions $f(x)$ exist we must have $f(0) = 1$.
Putting $u = v = x/2 \in \mathbb{R}$ again, but with $f(0) = 1$, we have

$$\begin{aligned} f(2u)f(0) &= 2u + f(0) \\ f(x) &= x + 1. \end{aligned}$$

Now we check this definition of f with the given relation.

$$\begin{aligned} f(u+v)f(u-v) &= (u+v+1)(u-v+1) \\ &= (u+1+v)(u+1-v) \\ &= (u+1)^2 - v^2 \\ &= u^2 + 2u + 1 - v^2 \\ &= 2u + (u^2 - v^2 + 1) = 2u + f(u^2 - v^2). \end{aligned}$$

So the definition $f(x) = x+1$ is valid and we have shown that there is exactly one function that satisfies the given equation.

Alternative Method. Start by setting $u = 0$ and $v = 1$. Then

$$\begin{aligned} f(1)f(-1) &= f(-1) \\ f(-1)\big(f(1) - 1\big) &= 0. \end{aligned}$$

Thus either $f(-1) = 0$ or $f(1) = 1$.
Suppose $f(1) = 1$ and let $u = 1$ and $v = 0$. Then

$$\begin{aligned} f(1)^2 &= 2 + f(1) \\ 1 &= 3, \qquad\qquad \text{(contradiction).} \end{aligned}$$

Thus, $f(1) \neq 1$. Hence, if functions $f(x)$ exist we must have $f(-1) = 0$.
Putting $u = v = x/2 \in \mathbb{R}$, we obtain

$$f(x)f(0) = x + f(0).$$

For $x = -1$, we have

$$\begin{aligned} 0 &= -1 + f(0) \\ f(0) &= 1 \\ \therefore f(x) &= x + 1 \end{aligned}$$

as before, and then we check this definition of f in the same way as before.

3. Let x be a non-zero real number such that $x + \dfrac{1}{x} \in \mathbb{Z}$. Prove $x^n + \dfrac{1}{x^n} \in \mathbb{Z}$, for all $n \in \mathbb{N}$.

Solution. Define

$$P(n) : f(n) \in \mathbb{Z},\ 0 \neq x \in \mathbb{R}, \qquad \text{where } f(n) = x^n + \frac{1}{x^n}.$$

We will prove the result by induction. We are given

$$x + \frac{1}{x} \in \mathbb{Z},\ 0 \neq x \in \mathbb{R},$$

i.e. $f(1) \in \mathbb{Z}$, which is $P(1)$. Thus $P(1)$ holds.
Now we attempt to show $P(k) \implies P(k+1)$ (this will amount to a false start, but it gives us insight into how to make a correct start). Thus we assume $P(k)$, i.e. that $f(k) = x^k + 1/x^k \in \mathbb{Z}$, and we also have $f(1) = x + 1/x \in \mathbb{Z}$. It follows that $f(k)f(1) \in \mathbb{Z}$. Now,

$$\begin{aligned} f(k)f(1) &= \Big(x^k + \frac{1}{x^k}\Big)\Big(x + \frac{1}{x}\Big) \\ &= x^{k+1} + \frac{1}{x^{k+1}} + x^{k-1} + \frac{1}{x^{k-1}} \\ &= f(k+1) + f(k-1). \end{aligned}$$

Thus

$$f(k+1) = f(k)f(1) - f(k-1).$$

So we see that $f(k+1) \in \mathbb{Z}$ only if we also have $f(k-1) \in \mathbb{Z}$, which is $P(k-1)$. Thus in fact,

$$P(k-1), P(k)[\text{ and } P(1)] \implies P(k+1).$$

This means to deduce that the next step of the induction holds we need the *two* previous steps, which means we must also show $P(2)$. Thus we now know how to proceed. Lets start again!

- $P(1)$ holds (given).
- Show $P(2)$ holds:

$$\begin{aligned} f(2) &= x^2 + \frac{1}{x^2} \\ &= x^2 + 2 \cdot x \cdot \frac{1}{x} + \frac{1}{x^2} - 2 \\ &= \Big(x + \frac{1}{x}\Big)^2 - 2 \\ &= f(1)^2 - 2 \in \mathbb{Z}, \qquad \text{since } f(1) \in \mathbb{Z} \text{ (given).} \end{aligned}$$

  Thus, $P(2)$ holds.
- Show $P(k-1), P(k) \implies P(k+1)$:
  We assume $P(k-1)$ and $P(k)$ hold. Then

$$\begin{aligned} f(k+1) &= f(k)f(1) - f(k-1) \\ &\in \mathbb{Z}, \qquad \text{since } f(1) \in \mathbb{Z} \text{ (given)}, \\ &\qquad\qquad\quad f(k-1) \in \mathbb{Z} \text{ (assumed } P(k-1) \text{ holds)} \\ &\qquad\qquad\quad f(k) \in \mathbb{Z} \text{ (assumed } P(k) \text{ holds).} \end{aligned}$$

  Thus $P(k+1)$ holds, if both $P(k-1)$ and $P(k)$ hold.

Thus, finally, invoking the Principle of Mathematical Induction, $P(n)$ holds for all $n \in \mathbb{N}$.

Hence, if $x + 1/x \in \mathbb{Z}$ for $0 \neq x \in \mathbb{R}$, then

$$x^n + \frac{1}{x^n} \in \mathbb{Z},\ \forall n \in \mathbb{N}.$$

4. The sequence $a_0, a_1, a_2, \ldots, a_{1997}$ has the properties:

   (i) $0 \le a_n \le 1$ for all $0 \le n \le 1997$,

   (ii) $a_n \ge \dfrac{a_{n-1} + a_{n+1}}{2}$ for all $1 \le n \le 1996$.

   (a) Prove that $a_{1997} - a_{1996} \le \dfrac{1}{1997}$.

   Solution. Rearranging (ii), we have

$$\begin{aligned} a_n &\ge \frac{a_{n-1} + a_{n+1}}{2} \\ 2a_n &\ge a_{n-1} + a_{n+1} \\ a_n - a_{n-1} &\ge a_{n+1} - a_n \\ a_{n+1} - a_n &\le a_n - a_{n-1} \text{ for } 1 \le n \le 1996. \end{aligned}$$

   Suppose, for a contradiction, that $a_{1997} - a_{1996} > \frac{1}{1997}$. Then

$$\begin{aligned} \frac{1}{1997} &< a_{1997} - a_{1996} \le a_{1996} - a_{1995} \le \cdots \le a_1 - a_0 && (1) \\ 1997 \cdot \frac{1}{1997} &< (a_{1997} - a_{1996}) + (a_{1996} - a_{1995}) + \cdots + (a_1 - a_0) && (2) \\ 1 &< a_{1997} - a_0 && (3) \\ a_0 &< a_{1997} - 1 \le 1 - 1, \qquad \text{by (i)} && (4) \\ a_0 &< 0, \qquad \text{contradicting (i).} && (5) \end{aligned}$$

   Therefore $a_{1997} - a_{1996} \not> \frac{1}{1997}$. Hence $a_{1997} - a_{1996} \le \frac{1}{1997}$.

   (b) Find a sequence satisfying (i) and (ii) such that $a_{1997} - a_{1996} = \dfrac{1}{1997}$.

   Solution. This changes the argument for (a), by replacing the '$<$' in (1) by '$=$', and the '$<$' of (2)–(5) by '$\le$'. In particular, we have

$$a_0 \le 0.$$

   But (i) implies $a_0 \ge 0$. Thus $a_0 = 0$ and equality is forced throughout (1)–(5), and we have

$$a_0 = 1, \quad \frac{1}{1997} = a_1 - a_0 = a_2 - a_1 = \cdots = a_{1997} - a_{1996}, \quad a_{1997} = 1,$$

   so that

$$a_n = \frac{n}{1997}, \text{ for } 0 \le n \le 1997,$$

   defines the unique sequence satisfying (i) and (ii) such that $a_{1997} - a_{1996} = \dfrac{1}{1997}$.

5. Let ABC be an acute-angled triangle with $\angle ACB = 60^\circ$. Let h_a be an altitude through A and let h_b be an altitude through B. Prove that the circumcentre of $\triangle ABC$ lies on the bisector of one of the four angles formed by h_a and h_b.