The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

1997 Senior Mathematics Contest Problems

1. On Planet Rhinochromos, 19971997 male monsters are to be married to the same number of female monsters. The same number of male monsters as females have purple noses; the rest have beige noses. The matching of males and females is performed randomly by Rhinochromos government monster psychologists.

   Show that the number of mixed marriages, i.e. marriages of partners with different nose colours is even.

2. Let c be a circle, A a point on c, and B and C two different points on c such that the chord BC is parallel to the tangent to c through A. Let P be on BC and let AP (extended) intersect c again at Q. Let k be the circle which touches BC at P and passes through Q.

   Prove that k touches c at Q.

3. Determine all integer pairs (x, y) that satisfy the equation
$$(x+1)^4 - (x-1)^4 = y^3.$$

4. Let c be a circle and let P be a point in the interior of c.
   (a) Let A, B, C, D be points on c such that the chords AC and BD intersect at right angles at P.
   Show that $AC^2 + BD^2$ is constant (i.e. the same for any such points A, B, C, D).
   (b) Amongst all such points A, B, C, D, show that $AC + BD$ attains its maximum value when $AC = BD$.

5. Let f be a function defined for all integers by
$$\begin{aligned} f(0) &= 1, \\ f(n) &= 0, && \text{for } -5 \le n \le -1, \\ f(n) &= f(n-6) + n, && \text{for all } n \in \mathbb{Z}. \end{aligned}$$

   Prove that
$$\frac{(n+1)(n+5)}{12} \le f(n) \le \frac{n^2 + 6n + 12}{12} \quad \text{for all } n \in \mathbb{Z}.$$