The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

1998 Senior Mathematics Contest Problems

1. Triangle $\triangle$ has side lengths a, b, c, triangle $\triangle_a$ has side lengths a', b, c, triangle $\triangle_b$ has side lengths a, b', c, triangle $\triangle_c$ has side lengths a, b, c', and each of the four triangles has area 1. Furthermore, $a \neq a'$, $b \neq b'$, $c \neq c'$.

   (a) Prove there exists a triangle $\triangle'$, with side lengths a', b', c'.

   (b) Determine the area of triangle $\triangle'$.

2. Determine all $n \in \mathbb{N}$ that satisfy

$$\sqrt{\frac{1+\frac{1}{2^{n-1}}}{2}} < 1 - \frac{2}{n}.$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ such that

   (i) $f(999 + x) = f(999 - x)$, and

   (ii) $f(1998 + x) = -f(1998 - x)$.

   Prove that f has the following two properties:

   (a) $f(-x) = -f(x)$ for all $x \in \mathbb{R}$, and

   (b) there exists $T \in \mathbb{R}$ such that $f(x + T) = f(x)$ for all $x \in \mathbb{R}$.

4. Let $ABCD$ be a cyclic quadrilateral with the property that its diagonals AC and BD intersect at right angles at M. Let N be the midpoint of AB, and let P be the point on CD such that NP and CD are perpendicular.

   Prove that M, N and P are collinear.

5. Let $n \in \mathbb{N}$. Prove

   (a) If there exists $a \in \mathbb{N}$ such that $a < n$ and the line defined by

$$\frac{x}{a} + \frac{y}{n-a} = 1$$

   contains a point, both of whose coordinates are positive integers, then n is not prime.

   (b) If n is not prime, then there exists $a \in \mathbb{N}$ such that $a < n$ and the line defined by

$$\frac{x}{a} + \frac{y}{n-a} = 1$$

   contains a point, both of whose coordinates are positive integers.