The University of Western Australia
DEPARTMENT OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

1999 Senior Mathematics Contest Problems

1. Circle k_1 has its centre on another circle k_2. The circles intersect at A and C. From any point B on k_2, draw BC, intersecting k_1 again at D.

   Prove that $AB = BD$.

2. Let $x, y, z \in \mathbb{Z}$ such that $\gcd(x, y, z) = 1$ and $x^2 + y^2 = z^2$.

   Prove that exactly one of x, y, z is divisible by 5.

3. Let $a_0, a_1, a_2 \in \mathbb{R}$ such that

   $$-1 \leq a_0 + a_1 x + a_2 x^2 \leq 1$$

   holds for all $x \in \mathbb{R}$ such that $-1 \leq x \leq 1$.

   Prove that

   $$-2 \leq a_2 \leq 2.$$

4. Let A, B, C, D, E be points in the (x, y)-plane, whose coordinates are integers.

   Prove that among the line segments joining these points there is at least one with a midpoint whose coordinates are integers.

5. Let $ABCD$ be a cyclic quadrilateral whose diagonals intersect in a right angle at E. Let U, V, W, Z be on AB, BC, CD, DA, respectively, such that $EU \perp AB$, $EV \perp BC$, $EW \perp CD$, $EZ \perp DA$.

   Prove that the quadrilateral $UVWZ$ is cyclic.