The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

2000 Senior Mathematics Contest Problems

1. Let $a, n \in \mathbb{N}$. Prove
$$\frac{(a+1)^{2n+1} + a^{n+2}}{a(a+1)+1} \in \mathbb{N}.$$

2. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x) + xf(1-x) = x^2 - 1,$$
holds for all $x \in \mathbb{R}$.

3. Prove that there do not exist $x, m, n \in \mathbb{N}$ such that
   (i) $m > 1$,
   (ii) $n > 1$, and
   (iii) $x^n + 1 = 2^m$.

4. Prove that
$$\frac{n}{2^1(n-1)} + \frac{n}{2^2(n-2)} + \cdots + \frac{n}{2^{n-2} \cdot 2} + \frac{n}{2^{n-1} \cdot 1} < 4,$$
for all integers $n > 1$.

5. Let $ABCD$ be a convex quadrilateral. Suppose the incircles of $\triangle ABC$ and $\triangle ACD$ touch each other.
   Prove that the incircles of $\triangle BDA$ and $\triangle BCD$ also touch each other.