The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

2001 Senior Mathematics Contest Problems

1. Prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ such that for all $r \in \mathbb{R}$,

   (i) $f(r^2) - \big(f(r)\big)^2 \geq \frac{1}{4}$, and

   (ii) the equation $f(x) = r$ has at most one solution.

2. In a certain village there are m houses around a circular pond, where $m > 1$ is odd. In each house live exactly n people.

   Show that in the village there is either a female with at least n female neighbours or a male with at least n male neighbours.

3. The real numbers $\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots$ satisfy

   (i) $a_1 = 1$, and

   (ii) $a_{m+n} - mn = a_m + a_n + 1$ for all $m, n \in \mathbb{Z}$.

   Find a formula for a_n in terms of n (for all integers n).

4. In $\triangle ABC$, let P_A, P_B, P_C be points on BC, CA, AB (extended if necessary), respectively. Let ℓ_A, ℓ_B, ℓ_C be lines in the plane of $\triangle ABC$ through P_A, P_B, P_C, respectively, such that $\ell_A \perp BC$, $\ell_B \perp CA$, $\ell_C \perp AB$.

   Prove that ℓ_A, ℓ_B, ℓ_C are concurrent if and only if

$$P_AC^2 + P_BA^2 + P_CB^2 = P_AB^2 + P_BC^2 + P_CA^2.$$

5. Prove that there is no pair of integers (x, y) other than the pairs $(0, 0)$ and $(0, -1)$ that satisfies the equation

$$y + y^2 = x + x^2 + x^3.$$