MATHEMATICS OLYMPIAD TRAINING SESSIONS

2002 Senior Mathematics Contest Problems with Some Solutions

1. Find all solutions of the following system of equations:

$$\frac{x_1}{x_1+1} = \frac{x_2}{x_2+3} = \frac{x_3}{x_3+5} = \cdots = \frac{x_{1001}}{x_{1001}+2001} \tag{1}$$

$$x_1 + x_2 + \cdots + x_{1001} = 2002 \tag{2}$$

Solution. First observe that

$$\begin{aligned} x_i = 0 \text{ for some } i &\implies x_i = 0 \text{ for all } i, \text{ by (1)} \\ &\implies x_1 + x_2 + \cdots + x_{1001} = 0, \text{ contradicting (2)} \\ \therefore\ x_i \neq 0 \text{ for all } i. \end{aligned}$$

Therefore, the reciprocals of the expressions in (1) are equal, i.e.

$$\begin{aligned}
(1) &\implies \frac{x_1+1}{x_1} = \frac{x_2+3}{x_2} = \cdots = \frac{x_{1001}+2001}{x_{1001}} \\
&\implies \frac{1}{x_1} = \frac{3}{x_2} = \cdots = \frac{2001}{x_{1001}}, \text{ subtracting 1 from each equated expression} \\
&\implies \quad x_2 = 3x_1 \\
&\qquad\quad x_3 = 5x_1 \\
&\qquad\qquad \vdots \\
&\qquad\quad x_i = (2i-1)x_1 \\
&\qquad\qquad \vdots \\
&\qquad x_{1001} = 2001x_1 \\
&\implies \text{by (2),} \\
&\qquad 2002 = (1 + 3 + \cdots + 2001)x_1 \\
&\qquad\qquad = 1001^2 x_1, \text{ since sum of the first } n \text{ odd numbers is } n^2 \\
&\qquad\quad x_1 = 2/1001 \\
&\implies x_1 = \frac{2}{1001}, x_2 = \frac{6}{1001}, \ldots, x_i = \frac{2(2i-1)}{1001}, \ldots, x_{1001} = \frac{4002}{1001}.
\end{aligned}$$

And the solution is unique.

2. Determine all $x, y \in \mathbb{N}$ such that

$$x! + 24 = y^2 \tag{$*$}$$

Solution. Consider cases according to the values of x.

Case 1: $x \geqslant 6$. Then

$$\begin{aligned} x! = \cdots 3 \cdots 6 \cdots \ &\text{is divisible by } 3^2 \\ \text{but } 24 = 2^3 \cdot 3 \ &\text{is only divisible by } 3 \\ \implies &\ 3 \text{ is the highest power of 3 that divides LHS}(*) \\ \implies &\ \text{LHS}(*) \text{ is not a square } \lightning \end{aligned}$$

$\therefore\ x < 6$.

Case 2: $x = 1$. Then $1! + 24 = 25 = 5^2 \implies (1, 5)$ is solution.

Case 3: $x = 2$. Then $2! + 24 = 26$, (not square)$\lightning$

Case 4: $x = 3$. Then $3! + 24 = 30$, (not square)$\lightning$

Case 5: $x = 4$. Then $4! + 24 = 48$, (not square)$\lightning$

Case 6: $x = 5$. Then $5! + 24 = 144 = 12^2 \implies (5, 12)$ is solution.

Therefore, the only solutions $(x, y) \in \mathbb{N}^2$ of $(*)$ are: $(1, 5)$ and $(5, 12)$. $\square$

3. For each pair (k, ℓ) of integers, determine all infinite sequences of integers $a_1, a_2, a_3, \ldots$ in which the sum of every 28 consecutive numbers equals k and the sum of every 15 consecutive numbers equals ℓ.

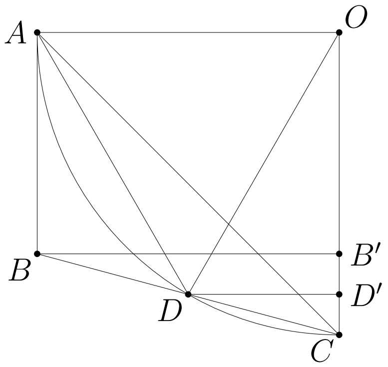
4. Determine all functions f that have the properties:

   (i) f is defined for all real numbers,

   (ii) $|f(x)| \leqslant 2002 \leqslant \left| \dfrac{xf(y) - yf(x)}{x - y} \right|$ for all x, y with $x \neq y$.

5. For $\triangle ABC$, let D be the midpoint of BC, $\angle BAD = \angle ACB$ and $\angle DAC = 15°$. Determine $\angle ACB$.

Solution.



$$\begin{aligned}
\text{Construct } K &= \text{circumcircle}(ADC)\\
\text{Then } \angle ACD &= \text{“angle on chord } AD \text{ in } K\text{”}\\
&= \text{“angle between } AD \text{ and } AB\text{”}\\
\therefore\ AB &\text{ is tangent to } K \text{ at } A, \text{ by}\\
&\ \text{(converse of) Tangent-Chord Th.}\\
\text{Let } O &= \text{centre}(K)\\
\text{Then } \angle DOC &= 2\angle DAC\\
&= 2 \cdot 15°\\
&= 30°\\
\text{Let } B', D' &\text{ be feet of perpendiculars}\\
&\ \text{dropped from } B, D \text{ resp. to } OC\\
\text{Then } \frac{DD'}{DO} &= \sin 30°\\
&= \tfrac{1}{2}\\
\therefore\ DD' &= \tfrac{1}{2}DO\\
BC &= 2DC\\
\therefore\ BB' &= 2DD'\\
&= DO\\
&= AO\\
\angle BB'O &= 90°\\
&= \angle BAO\\
BO &= OB\\
\therefore\ \triangle BB'O &\cong \triangle BAO, \text{ by RHS}\\
\therefore\ ABB'O &\text{ is a parallelogram with a rightangle}\\
\therefore\ ABB'O &\text{ is a rectangle}\\
\therefore\ \angle AOC &= 90°\\
\therefore\ \angle CAO &= \tfrac{1}{2}(180° - \angle AOC),\ \triangle AOC \text{ is isos.}\\
&= 45°\\
\angle BAC &= \angle BAO - \angle CAO\\
&= 45°\\
\therefore\ \angle ACB &= \angle BAD\\
&= \angle BAC - \angle DAC\\
&= 45° - 15°\\
&= 30°.
\end{aligned}$$