The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

2004 Senior Mathematics Contest Problems

1. Consider 8 points in a plane consisting of the 4 vertices of a square and the 4 midpoints of its edges. Each point is randomly coloured red, green, or blue with equal probability.

   Show that there is a more than 50% chance of obtaining a triangle whose vertices are 3 of these points coloured red.

2. Let $a_1, a_2, \ldots, a_{2004}$ be any non-negative real numbers such that
$$a_1 \geq a_2 \geq \cdots \geq a_{2004} \quad \text{and} \quad a_1 + a_2 + \cdots + a_{2004} \leq 1.$$
   Prove that
$$a_1^2 + 3a_2^2 + 5a_3^2 + 7a_4^2 + \cdots + 4007a_{2004}^2 \leq 1.$$

3. Let $f(n)$ be the integer closest to $\sqrt{n}$.

   Determine
$$\frac{1}{f(1)} + \frac{1}{f(2)} + \cdots + \frac{1}{f(10\,000)}.$$

4. Let $x, y, z \in \mathbb{N}$ such that $\dfrac{1}{x^2} + \dfrac{1}{y^2} = \dfrac{1}{z^2}$.

   Prove that $20 \mid xy$.

5. Let AB be the diameter of semicircle S, and let C and D be points on S other than A or B, with B closer to C than to D. Let AC and BD intersect in E and let AD (extended) and BC (extended) intersect in F. Let G and H be the midpoints of AE and BE, respectively, and let O be the circumcentre of $\triangle ABE$. Suppose that $DG \parallel CH$.

   Prove that $DG \parallel FO$.