The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

2005 Senior Mathematics Contest Problems

1. There are four coins placed on a table. Each of them has two sides, heads and tails. Here are three games you can play with the coins. In each game you have a positive integer n.

   (i) The game of "3-flip". At the start of the game, all four coins have heads up. A move in the game consists of turning over three coins at a time. The object of the game is to have all four coins tails up, after exactly n moves (no more, no less, and you can never pass).

   (ii) The game of "2-flip". At the start of the game, all four coins have heads up. A move consists of turning over two coins at a time. The object of the game is to have all four coins tails up, after exactly n moves (no more, no less, and no passing).

   (iii) The game of "frustration". At the start of the game, three coins have heads up, and one has tails up. A move consists of turning over two coins at a time. The object of the game is to have all four coins tails up, after exactly n moves (no more, no less, and no passing).

   Prove that you can win "3-flip" whenever $n \geq 4$ and n is even. Prove that you can win "2-flip" whenever $n \geq 2$. Prove that you can not win "frustration", for any n.

2. In one plane, let $ABCD$, $AEBF$ and $CEGH$ be squares, all of which have counter-clockwise orientation.

   Prove that B lies on the line segment DG and that $DB = BG$.

   Note. The square $PQRS$ is said to have *counter-clockwise orientation* if the interior of the square is to your left while you trace around the edges, in the order $P \to Q \to R \to S \to P$.

3. Determine a positive integer N such that for all positive integers m and n, with $m < n$ and $n \geq N$, the inequality
$$m < \left(1 + \frac{1}{2005}\right)^n$$
   is satisfied.

4. Determine all functions $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

   (i) $f(n, n) = n$,

   (ii) $f(m, n) = f(n, m)$, and

   (iii) if $n > m$, then $(n - m)f(m, n) = nf(m, n - m)$.

5. In a plane, let K_1 and K_2 be circles with centres O_1 and O_2, respectively, that intersect in points P and Q. Suppose the tangents of the two circles at P intersect in a right angle. A line through P intersects K_1 in A and K_2 in B. Let M be the midpoint of AB.

   Prove that $\angle O_1MO_2 = 90°$.