The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

2006 Senior Mathematics Contest: Solutions to Problems 4 and 5

4. Triangle ABC has a right angle at C. Suppose that D is the point on AB such that CD is perpendicular to AB. Let r_1, r_2 and r be the radii of the incircles of triangles ACD, BCD and ABC, respectively.

   Prove that $r_1 + r_2 + r = CD$.

Solution. The problem involves the inradii of three rightangled triangles. We first prove a lemma that relates the inradius of a rightangled triangle to the lengths of its sides.

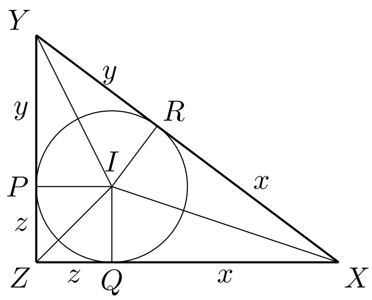
Lemma. *If XYZ is a rightangled triangle with rightangle at Z, then its inradius is given by*

$$\frac{XZ + YZ - XY}{2}.$$

Proof. Let the incircle of $\triangle XYZ$ have incentre I and touch the sides YZ, ZX and XY at P, Q and R, respectively.

By the RHS Rule (with Rightangle at P, Q or R where the sides of $\triangle XYZ$ are tangent to the incircle, Hypotenuse common, and Side an inradius) we have the congruences

$$\begin{aligned} \triangle IPZ &\cong \triangle IQZ \\ \triangle IQX &\cong \triangle IRX \\ \triangle IRY &\cong \triangle IPY. \end{aligned}$$



Hence we have $PZ = QZ = z$, $QX = RX = x$ and $RY = PY = y$, as per the diagram.

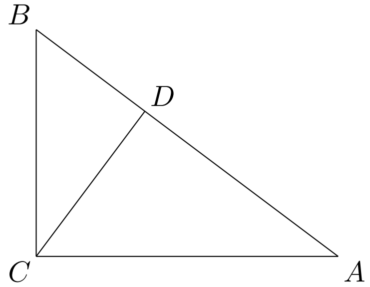
Moreover, since there are rightangles at each of P, Z and Q and two adjacent sides are equal, $PZQI$ is a square. Hence the inradius $r = z$, i.e.

$$r = z = \frac{(z+x) + (z+y) - (x+y)}{2} = \frac{XZ + YZ - XY}{2}.$$

□

Now we apply the lemma to the three triangles ACD, BCD and ABC of our problem and obtain

$$\begin{aligned} r_1 &= \tfrac{1}{2}(AD + CD - AC) \\ r_2 &= \tfrac{1}{2}(BD + CD - BC) \\ r &= \tfrac{1}{2}(AC + BC - AB) \\ \therefore r_1 + r_2 + r &= \tfrac{1}{2}(AD + BD - AB + 2CD) \\ &= CD \end{aligned}$$



5. Let $a_3, a_4, \ldots, a_{2005}, a_{2006}$ be real numbers with $a_{2006} \neq 0$.

Prove that there are not more than 2005 real numbers x such that

$$1 + x + x^2 + a_3x^3 + a_4x^4 + \cdots + a_{2005}x^{2005} + a_{2006}x^{2006} = 0.$$

Solution. It seems likely that there is nothing special about the number 2006 here, except perhaps that it's even. Define

$$p(x)1 + x + x^2 + a_3x^3 + \cdots + a_nx^n,$$

and assume that n is an even integer greater than 3. Let us prove that $p(x)$ does not have more than $n-1$ distinct real zeros.

For a contradiction, assume that $p(x)$ has n distinct real zeros, $x_1, x_2, \ldots, x_n$. Then

$$\begin{aligned} p(x) &= a_n(x - x_1)(x - x_2)\cdots(x - x_n) \\ &= a_n\bigl(x^n - (x_1 + x_2 + \cdots + x_n)x^n - 1 + \cdots + x_1x_2\cdots x_n\bigr) \end{aligned}$$

In general, the coefficient of $x^{n-\ell}$ in the expansion of $(x - x_1)(x - x_2)\cdots(x - x_n)$ is $(-1)^\ell$ times the sum of all possible products of the x_i taken ℓ at a time. Comparing coefficients with the first three of the given $p(x)$ we have

$$1 = a_nx_1x_2\cdots x_n \tag{1}$$

$$1 = -a_n\sum_{i=1}^{n}\Bigl(\prod_{\substack{1\le k\le n\\ k\ne i}} x_k\Bigr) \tag{2}$$

$$1 = a_n\sum_{1\le i<j\le n}\Bigl(\prod_{\substack{1\le k\le n\\ k\ne i, k\ne j}} x_k\Bigr) \tag{3}$$

Observe that (1) implies that none of the x_i is zero, so we can take out the righthand side of (1), which is 1, in (2) and (3) as follows.

$$1 = -a_nx_1x_2\cdots x_n\sum_{i=1}^{n}\frac{1}{x_i} \qquad = -\sum_{i=1}^{n}\frac{1}{x_i}, \qquad \text{using (2)} \tag{4}$$

$$1 = a_nx_1x_2\cdots x_n\sum_{1\le i<j\le n}\frac{1}{x_ix_j} = \sum_{1\le i<j\le n}\frac{1}{x_ix_j}, \quad \text{using (3).} \tag{5}$$

So we have from (4),

$$\begin{aligned} 1 &= \Bigl(-\sum_{i=1}^{n}\frac{1}{x_i}\Bigr)^2 \\ &= \sum_{i=1}^{n}\frac{1}{x_i^2} + 2\sum_{1\le i<j\le n}\frac{1}{x_ix_j} \\ &> 0 + 2\cdot 1 = 2, && \text{using (5).} \end{aligned}$$

But $1 > 2$ is absurd. Thus our original assumption is false, and so $p(x)$ cannot have n distinct real zeros. In particular, if $n = 2006$, then $p(x)$ has no more than 2005 real zeros.

It can be shown that a polynomial $p(x)$ of degree n with complex coefficients can be factorised into n linear factors over $\mathbb{C}$, which is to say $p(x)$ has n zeros (up to multiplicity) over $\mathbb{C}$. Further, if $p(x)$ has all real coefficients then any complex zeros come in conjugate pairs. So, in fact, the $p(x)$ of our problem has at most 2004 real zeros (up to multiplicity).