The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

2006 Senior Mathematics Contest: First 3 Problems with Solutions

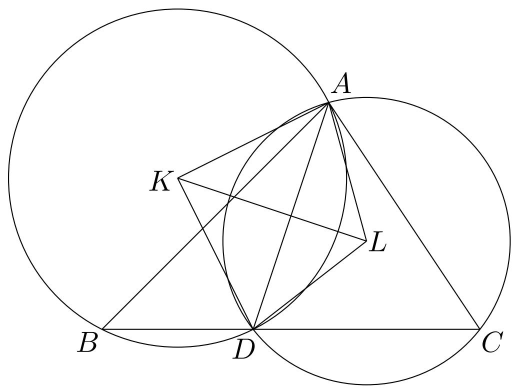
1. Let D be a point on side BC of triangle ABC. Let K, L be the circumcentres of triangles ABD and ADC, respectively.

   Prove that triangles ABC and AKL are similar.

   Solution. Firstly,

$$\begin{array}{rl} KA = KD, & \text{radii of circumcircle of } \triangle ABD \\ LA = LD, & \text{radii of circumcircle of } \triangle ADC \\ KL = KL, & \text{common side} \\ \therefore \triangle AKL \cong \triangle DKL, & \text{by the SSS Rule.} \end{array}$$

   Hence $\angle AKL = \angle DKL = \frac{1}{2}\angle AKD$ and $\angle ALK = \angle DLK = \frac{1}{2}\angle ALD$.



   We recall the result:

   > The angle subtended at a circle's centre is twice the angle subtended at the the circumference on the same arc.

$$\begin{array}{rl} \angle ABC = \angle ABD = \frac{1}{2}\angle AKD = \angle AKL, & \text{angles on arc } AD \text{ for circumcircle of } \triangle ABD \\ \angle ACB = \angle ACD = \frac{1}{2}\angle ALD = \angle ALK, & \text{angles on arc } AD \text{ for circumcircle of } \triangle ADC \\ \triangle ABC \sim \triangle AKL, & \text{by the AA Rule.} \end{array}$$

2. Prove that, among any fifteen composite numbers selected from the first 2006 positive integers, there will be two that are not relatively prime.

   Solution. Since $2006 < 47^2$, every composite number ≤ 2006 has a prime divisor < 47. There are precisely 14 primes < 47, namely

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43.$$

   Here are some intriguing statistics to remember. There are ...

   10 primes less than 30,
   15 primes less than 50, and
   25 primes less than 100.

   Check this for yourself!

   Hence, by the pigeonhole principle, at least two of a set of $15 = 14 + 1$ composites in $\{1, 2, \ldots, 2006\}$ are divisible by the same prime.

3. For each integer n, let a_n be the integer nearest to $\sqrt{n}$.
Prove that, for each positive integer n, the equation

$$a_1 + \cdots + a_{n^2+n} = 2(1^2 + \cdots + n^2)$$

holds.

Solution. We prove the result by induction. Define

$$P(n) : a_1 + \cdots + a_{n^2+n} = 2(1^2 + \cdots + n^2).$$

- For $n = 1$, $a_1 = a_2 = 1$. Note that $1^2 + 1 = 2$. So

$$\begin{aligned} \text{LHS of } P(1) = a_1 + a_2 &= 1 + 1 \\ &= 2(1^2) = \text{RHS of } P(1) \end{aligned}$$

  So $P(1)$ holds.
- Now we show $P(k) \implies P(k+1)$. The extra terms on the LHS of $P(k+1)$ are: $a_{k^2+k+1}, \ldots, a_{(k+1)^2+(k+1)}$. Observe that:

$$\begin{aligned} (k + \tfrac{1}{2})^2 &= k^2 + k + \tfrac{1}{4} < k^2 + k + 1 \\ \therefore k + \tfrac{1}{2} &< a_{k^2+k+1} \\ \therefore k + 1 &\le a_{k^2+k+1} \\ \big((k+1) + \tfrac{1}{2}\big)^2 &= (k+1)^2 + (k+1) + \tfrac{1}{4} > (k+1)^2 + (k+1) \\ \therefore a_{(k+1)^2+(k+1)} &< (k+1) + \tfrac{1}{2} \\ \therefore a_{(k+1)^2+(k+1)} &\le k + 1 \end{aligned}$$

  Also, $a_{k^2+k+1} \le a_{k^2+k+2} \le \cdots \le a_{(k+1)^2+(k+1)}$. So, in fact all these terms are equal to $k+1$, i.e.

$$a_{k^2+k+1} = a_{k^2+k+2} = \cdots = a_{(k+1)^2+(k+1)} = k + 1.$$

  So, assume $P(k)$. Then, we have

$$\begin{aligned} \text{LHS of } P(k+1) &= \sum_{i=1}^{(k+1)^2+(k+1)} a_k \\ &= \sum_{i=1}^{k^2+k} a_k + \sum_{i=k^2+k+1}^{(k+1)^2+(k+1)} a_k \\ &= 2(1^2 + 2^2 + \cdots + k^2) + \sum_{i=k^2+k+1}^{(k+1)^2+(k+1)} (k+1), \\ &\qquad \text{since } \sum_{i=1}^{k^2+k} a_k = \text{LHS of } P(k) = \text{RHS of } P(k) \text{ (inductive hypothesis)} \\ &= 2(1^2 + 2^2 + \cdots + k^2) + \big((k+1)^2 + (k+1) - (k^2 + k + 1) + 1\big)(k+1) \\ &= 2(1^2 + 2^2 + \cdots + k^2) + (2k+2)(k+1) \\ &= 2(1^2 + 2^2 + \cdots + k^2) + 2(k+1)^2 \\ &= 2\big(1^2 + 2^2 + \cdots + k^2 + (k+1)^2\big) \\ &= \text{RHS of } P(k+1) \end{aligned}$$

  So we have shown that $P(k) \implies P(k+1)$ for $k \in \mathbb{N}$.

So, by induction, we have $P(n)$ holds for all $n \in \mathbb{N}$, i.e.

$$a_1 + \cdots + a_{n^2+n} = 2(1^2 + \cdots + n^2), \text{ for all } n \in \mathbb{N}.$$