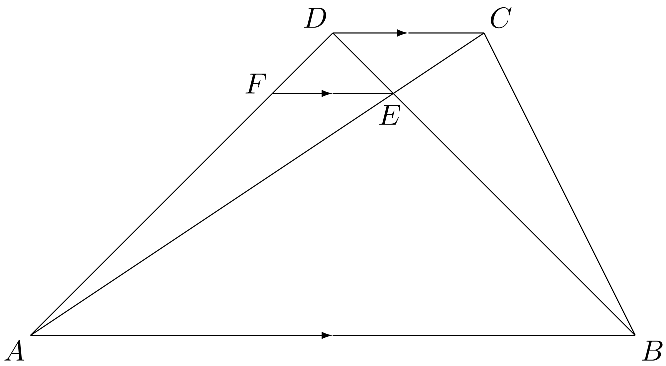
The University of Western Australia
SCHOOL OF MATHEMATICS & STATISTICS

AMO TRAINING SESSIONS

2007 Senior Mathematics Contest: First 3 Problems with Solutions

1. Let $ABCD$ be a trapezium, where sides AB and CD are parallel. Its diagonals AC and BD intersect in E. Further, let F be the point on AD such that EF and AB are parallel. Prove that $AF \times BD \times CE = AC \times DF \times BE$.

Solution.



Since parallel lines cut transversals in equal ratios we have

$$AF : FD = BE : ED = AE : EC = x : y \text{ say.}$$

Thus we have

$$\begin{aligned}
AF &= \frac{x}{x+y}AD && (1)\\
DF &= \frac{y}{x+y}AD && (2)\\
BE &= \frac{x}{x+y}BD && (3)\\
DE &= \frac{y}{x+y}BD\\
AE &= \frac{x}{x+y}AC\\
CE &= \frac{y}{x+y}AC && (4)
\end{aligned}$$

Thus we have

$$\begin{aligned}
AF \times BD \times CE &= \frac{x}{x+y}AD \times \frac{x+y}{x}BE \times \frac{y}{x+y}AC, && \text{using (1), (3) and (4)}\\
&= \frac{x}{x+y}\,\frac{x+y}{y}DF \times \frac{x+y}{x}BE \times \frac{y}{x+y}AC, && \text{using (2)}\\
&= DF \times BE \times AC\\
&= AC \times DF \times BE
\end{aligned}$$

2. Determine all triples (x, y, p) of integers such that $x^4 + 4y^4 = p$ and p is a prime number.

Solution.

$$\begin{aligned} x^4 + 4y^4 &= (x^2 + 2y^2)^2 - 2x^2 \cdot 2y^2 \\ &= (x^2 + 2y^2)^2 - (2xy)^2 \\ &= \big((x^2 + 2y^2) - 2xy\big)\big((x^2 + 2y^2) + 2xy\big) \\ &= \big((x - y)^2 + y^2\big)\big((x + y)^2 + y^2\big) \end{aligned}$$

Since $x, y \in \mathbb{Z}$ and $x^4 + 4y^4 = p$ prime, one of $(x - y)^2 + y^2$, $(x + y)^2 + y^2$ is 1 and the other is p.

Case 1: $(x - y)^2 + y^2 = 1$. We have a sum of integer squares that is 1; so one of $(x - y)^2$, y^2 is 0 and the other is 1.
If $(x-y)^2 = 1$ and $y^2 = 0$, we have $x = \pm 1$, $y = 0$, in which case $(x+y)^2+y^2 = 1$, also, and is not prime. Hence we get no solutions in this instance.
If $(x - y)^2 = 0$ and $y^2 = 1$, we have $x = y = \pm 1$, in which case

$$(x + y)^2 + y^2 = (\pm 2)^2 + (\pm 1)^2 = 5,$$

which is prime, giving solutions $(x, y, p) = (1, 1, 5)$ or $(-1, -1, 5)$.

Case 2: $(x + y)^2 + y^2 = 1$ (and $(x - y)^2 + y^2 = p$). This is just Case 1 with y replaced by $-y$. So the solutions here are $(x, y, p) = (1, -1, 5)$ or $(-1, 1, 5)$.

Thus we have altogether 4 solutions:

$$(x, y, p) = (1, 1, 5), \quad (-1, -1, 5), \quad (1, -1, 5), \quad (-1, 1, 5).$$

3. For each positive integer n, let $f(n)$ be the smallest of all positive integers k such that n divides $k!$.

Determine the largest value of $\dfrac{f(n)}{n}$ as n runs over all composite numbers greater than 4.

Solution. We are given that n is composite and greater than 4.

Case 1: $n = ab$, $a, b \in \mathbb{N}$ with $1 < a < b < n$. Here $b! = 1 \cdots a \cdots b$ is divisible by n.
$\therefore f(n) \le b$, so that

$$\frac{f(n)}{n} = \frac{f(n)}{ab} \le \frac{b}{ab} = \frac{1}{a} \le \frac{1}{2}.$$

Case 2: $n = p^2$ for some prime $p > 2$. We require $p^2 \mid k!$. The only factors of $k!$ divisible by p are multiples of p, namely $p, 2p, \dots$.
So $p \,|\, p!$ put $p^2 \nmid p!$.
Thus $k = (2p)! = 1 \cdot 2 \cdots p \cdots (2p)$ is the least k for which $p \,|\, k!$.
$\therefore f(n) = 2p$, and hence

$$\frac{f(n)}{n} = \frac{2p}{p^2} = \frac{2}{p} \le \frac{2}{3}.$$

For $p = 3$, i.e. $n = 9$ we obtain $f(n)/n = \frac{2}{3}$.

Observe, that there is no need to refine the bound in Case 1. since the bound in Case 2. is larger and attained when $n = 9$.
Thus the largest value of $f(n)/n$ is $\frac{2}{3}$.